PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 1, November 1973

# COMMON FIXED POINTS FOR SEMIGROUPS OF MAPPINGS

ANTHONY T. LAU AND CHI SONG WONG[1]

ABSTRACT. Let $X$ be a compact convex subset of a strictly convex Banach space. Let $S$ be a Hausdorff topological semigroup which is either left amenable or left reversible. Then for any generalised nonexpansive (jointly) continuous action of $S$ on $X$, $X$ contains a common fixed point of $S$.

1. **Introduction.** Let $S$ be a (nonempty) *topological semigroup*, i.e. $S$ is a semigroup with a Hausdorff topology such that for each $s$ in $S$, the mappings $x \to sx$ and $x \to xs$ of $S$ into $S$ are continuous. Let $C(S)$ be the Banach algebra of all bounded real-valued continuous functions on $S$ with the supremum norm. A function $f$ in $C(S)$ is *left uniformly continuous* if the mapping $s \to l_s f$, where $l_s f(t) = f(st)$ for all $s, t \in S$, is continuous on $S$ ([**12**], [**13**]). Let LUC($S$) be the family of all left uniformly continuous functions on $S$. Then LUC($S$) is a Banach subalgebra of $C(S)$ which contains all of the real-valued constant functions on $S$ and is translation invariant [**13**]. $S$ is *left amenable* if LUC($S$) has a left *invariant mean* $\mu$, i.e. $\mu$ is a continuous linear functional on LUC($S$) such that $\|\mu\| = \mu(e) = 1$ and $\mu(l_s f) = \mu(f)$ for all $s \in S, f \in$ LUC($S$), where $e$ is the function with $e(s) = 1$ for all $s \in S$ [**13**]. When $S$ is discrete, this definition coincides with that of M. M. Day in [**1**].

Let $X$ be a subset of a Banach space $B$ with norm $p$. An *action* of $S$ on $X$ is a mapping of $S \times X$ into $X$, denoted by $(s, x) \to sx$ such that $(st)x = s(tx)$ for all $s, t \in S$, $x \in X$ (as a consequence, $S$ can be considered as a family of functions of $X$ into $X$ with the possibility that different elements in $S$ correspond to the same function). A point $x$ in $X$ is a common fixed point of $S$ (with respect to an action) if $sx = x$ for all $s \in S$. An action

Received by the editors March 2, 1973.

*AMS (MOS) subject classifications* (1970). Primary 47H10; Secondary 54H25.

*Key words and phrases.* Common fixed points, equicontinuous mapping, generalized nonexpansive mapping, left amenable topological semigroup, left reversible topological semigroup, strictly convex Banach space, strongly almost periodic function.

[1] The first author is supported by NRC Grant A7679. The second author is partially supported by NRC Grant A8518 and a grant from the Mathematical Congress. The first author would like to thank Professor T. Rogers for providing a preprint of his joint paper [**6**] with G. Hardy.

of $S$ on $X$ is *nonexpansive* if $p(sx-sy) \leqq p(x-y)$ for all $s \in S$, $x, y \in X$; it is *generalized nonexpansive* if there exist nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ such that $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=1$ and for all $s \in S$, $x, y \in X$,

$$p(sx - sy) \leqq \alpha_1 p(x - sx) + \alpha_2 p(y - sy) + \alpha_3 p(x - sy) + \alpha_4 p(y - sx) + \alpha_5 p(x - y).$$

It is obvious that an action of $S$ on $X$ is generalized nonexpansive if it is nonexpansive. The converse is not true even for the case when $X$ is a bounded closed interval [**17**].

A Banach space $B$ with norm $p$ is *strictly convex* if for any $x, y, z$ in $B$, $p(x-z)+p(z-y)=p(x-y)$ implies that $z \in [x, y]$ $(=\{(1-t)x+ty: t \in [0, 1]\})$. It is the main purpose of this paper to prove the following result. Related results for family of nonexpansive mappings can be found in [**3**], [**7**], [**11**], [**15**] and [**16**]. The notion of generalized nonexpansive mappings for metric spaces are considered in [**4**], [**8**], [**14**] and [**6**].

THEOREM 1. *Let $X$ be a compact convex subset of a strictly convex Banach space $B$ with norm $p$. Let $S$ be a left amenable topological semigroup. Then for any (jointly) continuous generalized nonexpansive action of $S$ on $X$, $S$ has a common fixed point in $X$.*

2. **Proof of Theorem 1.** Since the given action is generalized nonexpansive there exist nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ such that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1 \tag{1}$$

and for all $s \in S$, $x, y \in X$,

$$p(sx - sy) \leqq \alpha_1 p(x - sx) + \alpha_2 p(y - sy) + \alpha_3 p(x - sy) + \alpha_4 p(y - sx) + \alpha_5 p(x - y). \tag{2}$$

By calculating $(p(sx-sy)+p(sy-sx))/2$ through (2), we may without loss of generality assume that $\alpha_1=\alpha_2$ and $\alpha_3=\alpha_4$. Thus $\alpha_1=\alpha_2 \leqq \frac{1}{2}$, $\alpha_3=\alpha_4 \leqq \frac{1}{2}$.

For simplicity, a subset $Y$ of $X$ is said to be *invariant* if $sy \in Y$ for all $s \in S$, $y \in Y$. By Zorn's lemma, there exists a minimal nonempty closed convex invariant subset $C$ of $X$. Again by Zorn's lemma, there exists a minimal nonempty invariant closed subset $K$ of $C$. We shall first prove that $sK=K$ for all $s \in S$. Let $x_0 \in K$. $x_0$ will now be used to obtain a measure $\lambda$ on the $\sigma$-algebra $B(K)$ of all Borel subsets of $K$. For each $f$ in $B(K)$, let $Tf(s)=f(sx_0)$, $s \in S$. Then $Tf \in \mathrm{LUC}(S)$ [**12**, proof of Theorem 1]. Since $S$ is left amenable, LUC($S$) has a left invariant mean $\mu$. Let $\lambda=T^*\mu$, where $T^*$ is the adjoint of $T$. Then $\|\lambda\|=\lambda(e)=1$ and $\lambda({}_sf)=\lambda(f)$ for all

$f \in C(K)$, $s \in S$, where ${}_s f(y)=f(sy)$, $y \in K$. By the Riesz representation theorem, $\lambda$ can be considered as a measure on $B(K)$ with $\lambda(K)=1$ and $\lambda(s^{-1}(A))=\lambda(A)$ for all $s \in S$, $A \in B(K)$, where $s^{-1}(A)=\{y \in K : sy \in A\}$. Since $K$ is compact, the support supp $\lambda$ of $\lambda$ is the smallest compact subset $Y$ of $K$ with $\lambda(Y)=1$. Let $s \in S$. Then from $\lambda(s^{-1}(\operatorname{supp} \lambda))=\lambda(\operatorname{supp} \lambda)=1$, we have $s^{-1}(\operatorname{supp} \lambda) \supset \operatorname{supp} \lambda$. So $s(\operatorname{supp} \lambda) \subset s(s^{-1}(\operatorname{supp} \lambda)) \subset \operatorname{supp} \lambda$ and therefore supp $\lambda$ is invariant. By minimality of $K$, $\operatorname{supp} \lambda = K$. Since $\lambda(sK)=\lambda(s^{-1}(s(K))) \geqq \lambda(K)=1$, $sK \supset \operatorname{supp} \lambda = K$. Hence $sK=K$.

Now note that if $K$ is a singleton, then the point in $K$ is a common fixed point of $S$. So we may assume that $K$ contains at least two points.

*Case* 1. $\alpha_1=\alpha_2=0$. By Lemma 1 in [3], there exists $z_0 \in C$ such that

$$\sup\{p(z_0 - x) : x \in K\} \leqq r$$

for some $r<\delta(K)$ $(=\sup\{p(x-y) : x, y \in K\})$. Let

$$W = \{z \in C : p(z - x) \leqq r \text{ for all } x \in K\}.$$

Then $z_0 \in W$ and $W$ is a closed convex subset of $C$. To see that $W$ is invariant, let $z \in W$, $s \in S$. By compactness of $K$, there exists $y_1 \in K$ such that $p(sz-y_1)=\sup\{p(sz-x) : x \in K\}$. Since $sK=K$, $sy_2=y_1$ for some $y_2 \in K$. Now from (2)

$$\begin{aligned} p(sz - y_1) &= p(sz - sy_2) \\ &\leqq \alpha_3 p(z - y_1) + \alpha_4 p(sz - y_2) + \alpha_5 p(z - y_2) \\ &\leqq (\alpha_3 + \alpha_5) r + \alpha_4 p(sz - y_1). \end{aligned}$$

Since $1-\alpha_4 \geqq \frac{1}{2} > 0$,

$$p(sz - y_1) \leqq \frac{\alpha_3 + \alpha_5}{1 - \alpha_4} r = r.$$

By the choice of $y_1$, $sz \in W$. Thus $W$ is invariant. By minimality of $C$, $W=C$. Hence by definition of $W$, $\delta(K) \leqq r < \delta(K)$, a contradiction.

*Case* 2. $\alpha_1=\alpha_2 \neq 0$, $\alpha_3=\alpha_4 \neq 0$. Let $s \in S$. Then from $sK=K$ and (2),

$$\begin{aligned} \delta(K) &= \sup\{p(sx - sy) : x, y \in K\} \\ &\leqq \sup\{\alpha_1 p(x - sx) + \alpha_2 p(y - sy) + \alpha_3 p(x - sy) \\ &\qquad\qquad + \alpha_4 p(y - sx) + \alpha_5 p(x - y) : x, y \in K\} \\ &\leqq \alpha_1 \sup\{p(x - sx) : x \in K\} + \alpha_2 \sup\{p(y - sy) : y \in K\} \\ &\quad + \alpha_3 \sup\{p(x - sy) : x, y \in K\} + \alpha_4 \sup\{p(y - sx) : x, y \in K\} \\ &\quad + \alpha_5 \sup\{p(x - y) : x, y \in K\} \\ &\leqq \alpha_1 \delta(K) + \alpha_2 \delta(K) + \alpha_3 \delta(K) + \alpha_4 \delta(K) + \alpha_5 \delta(K) \\ &= \delta(K). \end{aligned} \tag{3}$$

Since $\alpha_1=\alpha_2\neq 0$, we have from (3),

$$\delta(K) = \sup\{p(x - sx):x \in K\}. \tag{4}$$

From (4), $sK=K$ and (2),

$$\begin{aligned}\delta(K) &= \sup\{p(sx - s^2x):x \in K\}\\ &\leqq \sup\{\alpha_1 p(x - sx) + \alpha_2 p(sx - s^2x)\\ &\qquad + \alpha_3 p(x - s^2x) + \alpha_5 p(x - sx):x \in K\}\\ &\leqq \alpha_1\delta(K) + \alpha_2\delta(K) + \alpha_3\delta(K) + \alpha_5\delta(K).\end{aligned}$$

Hence $1\leqq\alpha_1+\alpha_2+\alpha_3+\alpha_5<1$, a contradiction.

*Case* 3. $\alpha_1=\alpha_2\neq 0$, $\alpha_3=\alpha_4=0$. Let $s\in S$. By the Schauder-Tychonoff fixed point theorem, $sw=w$ for some $w\in C$. Since $K$ is compact, there exists $y_1\in K$ such that $p(y_1-w)=\sup\{p(x-w):x\in K\}$. Since $sK=K$, $y_1=sy_2$ for some $y_2\in K$. Thus

$$\begin{aligned}p(w - y_1) = p(sw - sy_2) &\leqq \alpha_2 p(y_2 - sy_2) + \alpha_5 p(w - y_2).\\ &\leqq \alpha_2 p(y_2 - sy_2) + \alpha_5 p(w - y_1).\end{aligned}$$

Since $1-\alpha_5=\alpha_1+\alpha_2>0$ and $\alpha_2/(1-\alpha_5)=\frac{1}{2}$,

$$p(w - y_1) \leqq \frac{\alpha_2}{1 - \alpha_5}\, p(y_2 - sy_2) \leqq \tfrac{1}{2}\delta(K).$$

By the choice of $y_1$,

$$p(w - x) \leqq \tfrac{1}{2}\delta(K) \quad \text{for all } x \in K. \tag{5}$$

By compactness of $K$, there exist $x_1, x_2\in K$ such that $\delta(K)=p(x_1-x_2)$. Now from (5), $\delta(K)=p(x_1-x_2)\leqq p(x_1-w)+p(w-x_2)\leqq\delta(K)$, i.e.

$$p(x_1 - x_2) = p(x_1 - w) + p(w - x_2). \tag{6}$$

Since $B$ is strictly convex, we have from (5) and (6), $w=\frac{1}{2}(x_1+x_2)$. Since $x_1$, $x_2$ do not depend on $w$, $(x_1+x_2)/2$ is a common fixed point of $S$.

3. **Related results.** Let $S$ be a topological semigroup. $S$ is left *reversible* if the family of all closed right ideals in $S$ has the finite intersection property. When $S$ has the discrete topology, $S$ is left reversible if it is left amenable [**5**, p. 181]. However, in general, "left reversible" and "left amenable" are two independent conditions on a topological semigroup ([**1**, p. 516], [7, §4]). By using Lemma 1 in [**9**], we still have $sK=K$ for all $s\in K$ even if, in Theorem 1, $S$ is left reversible instead of being left amenable. So we have the following result.

THEOREM 2. *Let $X$ be a compact convex subset of a strictly convex Banach space $B$ with norm $p$. Let $S$ be a left reversible topological semigroup.*

*Then for any (jointly) continuous generalized nonexpansive action of $S$ on $X$, $S$ has a common fixed point in $X$.*

Let $S$ be a topological semigroup. A function $f \in C(S)$ is *strongly almost periodic* if $\{l_s f : s \in S\}$ is relatively compact in $C(S)$. Let $\mathrm{AP}(S)$ be the family of all strongly almost periodic functions on $S$. Then $\mathrm{AP}(S)$ is a Banach subalgebra of $\mathrm{LUC}(S)$ which contains all constant real-valued functions on $S$ and is translation invariant [**2**]. An action of $S$ on a compact subset $X$ of a Banach space is *equicontinuous* if $S$ is equicontinuous when it is considered as a family of functions of $X$ into $X$. Now by using Lemma 3.1 in [**10**] and by modifying the proof of Theorem 1 in an obvious way, we have the following result.

THEOREM 3. *Let $X$ be a compact subset of a Banach space $B$ with norm $p$. Let $S$ be a topological semigroup such that* $\mathrm{AP}(S)$ *has a left invariant mean. Then for any equicontinuous and jointly continuous action of $S$ on $X$, $S$ has a common fixed point in $X$.*

We would like to point out here that by modifying the definitions and proofs in an obvious way, one can prove Theorems 1–3 for the case when $B$ is a Hausdorff locally convex topological space with its topology induced by a given family of pseudonorms on $B$.

## REFERENCES


**1.** M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544. MR **19**, 1067.

**2.** K. deLeeuw and I. Glickberg, *Application of almost periodic compactification*, Acta. Math. **105** (1961), 63–97. MR **24** #A1632.

**3.** R. DeMarr, *Common fixed points for commuting contraction mappings*, Pacific J. Math. **13** (1963), 1139–1141. MR **28** #2446.

**4.** M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79. MR **24** #A2936.

**5.** E. E. Granirer, *Extremely amenable semigroups*, Math. Scand. **17** (1965), 177–197. MR **33** #5760.

**6.** G. Hardy and T. Roger, *A generalisation of a fixed point theorem of S. Reich*, Canad. Math. Bull. (to appear).

**7.** R. D. Holmes and A. Lau, *Nonexpansive actions of topological semigroups and fixed points*, J. London Math. Soc. **5** (1972), 330–336.

**8.** R. Kannan, *Some results on fixed points*. II, Amer. Math. Monthly **76** (1969), 405–408. MR **41** #2487.

**9.** A. T. Lau, *Invariant means on subsemigroups of locally compact groups*, Rocky Mountain J. Math. **3** (1973), 77–81.

**10.** ———, *Invariant means on almost periodic functions and fixed point properties*, Rocky Mountain J. Math. **3** (1973), 69–76.

**11.** T. Mitchell, *Fixed points of reversible semigroups of nonexpansive maps*, Kodai Math. Sem. Rep. **22** (1970), 322–323. MR **42** #2316.

**12.** T. Mitchell, *Topological semigroups and fixed points*, Illinois J. Math. **14** (1970), 630–641. MR 42 #5245.

**13.** I. Namioka, *On certain actions of semi-groups on L-spaces*, Studia Math. **29** (1967), 63–77. MR **36** #6910.

**14.** S. Reich, *Some remarks concerning contractive mappings*, Canad. Math. Bull. **14** (1971), 121–124. MR **45** #1145.

**15.** W. Takahashi, *Fixed point theorem for amenable semigroup of nonexpansive mappings*, Kodai Math. Sem. Rep. **21** (1969), 383–386. MR **41** #7501.

**16.** Chi Song Wong, *Fixed point theorems on compact Hausdorff spaces* (to appear).

**17.** ———, *Fixed points of generalised nonexpansive mappings on an int*
Proc. Amer. Math. Soc. (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WINDSOR, WINDSOR, ONTARIO, CANADA