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REGULARLY VARYING SEQUENCES

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ABSTRACT. A simple necessary and sufficient condition is developed for a sequence $\{\theta(n)\}$, $n=0, 1, 2, \cdots$, of positive terms, to satisfy $\theta(n)=R(n)$, $n\geqq 0$, where $R(\cdot)$ is a regularly varying function on $[0, \infty)$. The condition (2.1), below, leads to a Karamata-type exponential representation for $\theta(n)$. Various associated difficulties are also discussed. (The results are of relevance in connection with limit theorems in various branches of probability theory.)

1. **Introduction.** A function $R(\cdot)$, defined, finite, positive and measurable on $[A, \infty)$ for some $A\geqq 0$, is said to be regularly varying if for each $\lambda>0$

$$\lim_{x\to\infty}\frac{R(\lambda x)}{R(x)} = \phi(\lambda) \tag{1.1}$$

where $0<\phi(\lambda)<\infty$. (In actual fact a weaker definition can be used, for the assumption that this positive finite limit property obtains for all λ in a subset of positive measure of $(0, \infty)$ implies that it obtains for all $\lambda \in (0, \infty)$.) Since $\phi(\lambda)$ is a positive measurable solution of the functional equation

$$\phi(uv) = \phi(u)\phi(v), \qquad u, v>0, \tag{1.2}$$

it is well known that $\phi(\lambda)=\lambda^\rho$ for some finite ρ, and so we can write $R(x)=x^\rho L(x)$ where

$$\lim_{x\to\infty}\frac{L(\lambda x)}{L(x)} = 1, \quad \text{for each } \lambda > 0; \tag{1.3}$$

such a regularly varying function, for which the index ρ of regular variation is zero, is called slowly varying.

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The two most important properties of regularly varying functions (from which others are easily deducible) are:

(i) The convergence in (1.1) (or, equivalently, (1.3)) is uniform for λ in any fixed interval $[a, b]$, $0<a<b<\infty$.

(ii) For some $B \geqq A$, a slowly varying function L has representation

$$L(x) = \exp\left\{\eta(x) + \int_B^x \frac{\varepsilon(t)}{t}\, dt\right\}, \qquad x \geqq B, \tag{1.4}$$

where $\eta(x) \to c$ ($|c| < \infty$) as $x \to \infty$ and $\varepsilon(t) \to 0$ as $t \to \infty$, both being measurable and bounded. Conversely, any function L having representation (1.4) is clearly slowly varying.

The systematic development of the notion of a regularly varying function, of great importance in probability theory, is due, for continuous functions, to Karamata ([1930], [1933]), and in the above setting to various later authors. A sketch of the basic history and theory is given in §§1 and 4 of the recent paper of Bojanić and Seneta [1971]. We pause to note only the result of de Bruijn [1959, §4] that in (1.4), $\varepsilon(t)$ may be taken as continuous (the less desirable properties of L being perpetuated by $\eta(x)$). This last remark enables us to deduce that as $x \to \infty$

$$R(x) = x^\rho L(x) \sim x^\rho L_1(x) = R_1(x)$$

where $(R_1 x) = x^\rho \exp\{c + \int_B^x \varepsilon(t)/t\, dt\}$, $x \geqq B$, is a continuously differentiable regularly varying function such that

$$xR_1'(x)/R_1(x) \to \rho \tag{1.5}$$

as $x \to \infty$, since

$$xR_1'(x)/R_1(x) = \rho + \varepsilon(x), \qquad x \geqq B. \tag{1.6}$$

Conversely, any function R_1 satisfying (1.5) is regularly varying (with index ρ), as can be seen by defining $\varepsilon(x)$ from (1.6) and integrating for R_1, to obtain the required representation.

More recently, a problem of the following *genre* has occurred in several probabilistic contexts. Given a sequence $\{\theta(n)\}$, $n=0, 1, 2, \cdots$, of positive numbers, when is it possible to imbed it in a regularly varying function? In other words, when is it possible to find a regularly varying function $R(x)$ such that $R(n)=\theta(n)$? If is it possible, then it follows, for example, from either property (i) or (ii) of regularly varying functions, that

$$\theta(n + 1)/\theta(n) \to 1 \tag{1.7}$$

as $n \to \infty$. As examples of results obtained so far, we mention that of de Haan [1970, pp. 6–8], who shows that the imbedding is possible if (a) $\{\theta(n)\}$ is monotone, and (b) $\theta(nm)/\theta(n) \to m^\rho$ for all positive m as $n \to \infty$, where ρ is finite, and that of R. S. Slack, which asserts that in (b), m^ρ may be replaced by $\phi(m)$, $0 < \phi(m) < \infty$, if (1.7) is imposed as an additional hypothesis, with the same conclusion.[2]

This type of problem, concerning regular behavior of sequences, was studied prior to the papers of Karamata mentioned above. The reader may want to consult the works of Schmidt [1925] and Schur [1930] in this regard.

There is some difficulty in attempting the obvious approach to the sequence problem along the lines of the elegant definition (1.1). Thus, it is possible to construct a sequence of positive numbers $\{\theta(n)\}$ satisfying simultaneously (for positive integer k), as $n \to \infty$,

$$\theta(nk)/\theta(n) \to 1, \qquad \theta(n+1)/\theta(n) \not\to 1, \tag{1.8}$$

so that the requirement (1.7) is broken.

To carry out such a construction, let $\omega(n)$ denote the number of prime divisors of n. Let $\theta(n) = \omega(n) + (\log\log n)^{1/2}$, $n \geqq 2$. It is known (Kubilius [1964, p. 39]) that there exists a subsequence $p_{j_1}, p_{j_2}, \cdots$ of the primes such that $\omega(p_{j_n} - 1) \sim \log\log p_{j_n}$ as $n \to \infty$. If we consider the subsequence $\theta(p_{j_n})/\theta(p_{j_n} - 1)$ of the sequence $\theta(n+1)/\theta(n)$, we readily see that its limit is zero, since $\omega(p_{j_n}) = 1$, so that $\theta(n+1)/\theta(n) \not\to 1$. If we consider $\theta^*(n) = \omega(n) + \log\log n$ instead, we obtain that $\theta^*(p_{j_n})/\theta^*(p_{j_n} - 1) \to \frac{1}{2}$ as $n \to \infty$, also a satisfactory result. On the other hand, we have for integer $k \geqq 1$

$$\theta(nk) = \omega(nk) + (\log(\log n + \log k))^{1/2},$$

whence for large n,

$$\begin{aligned}\omega(n) &+ (\log(\log n + \log k))^{1/2} \\ &\leqq \theta(nk) \leqq \omega(n) + \omega(k) + (\log(\log n + \log k))^{1/2}\end{aligned}$$

from the definition ω; and so $\theta(nk)/\theta(n) \to 1$, each integer $k \geqq 1$. (It may be proved similarly that $\theta^*(nk)/\theta^*(n) \to 1$.)

To conclude this section, it is necessary to mention that Ibragimov and Linnik [1971, p. 397] seem to cite, as an example of a sequence of positive terms such that $\theta(nk)/\theta(n) \to 1$ as $n \to \infty$, but $\theta(n+1)/\theta(n) \not\to 1$, the sequence given by $\theta(n) = \omega(n) + (\log n)^{1/2}$. Whereas it is easy to check that $\theta(nk)/\theta(n) \to 1$ for each positive integer k, the proposition regarding

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[2] Mentioned in a letter to one of the authors from G. E. H. Reuter. Slack's result occurs in a branching process context, and his method is not known to us. Nevertheless, one of us has constructed a possibly different proof.

$\theta(n+1)/\theta(n)$ appears to be a deeper one whose validity or otherwise is not known; but note that $\theta(p_{j_n})/\theta(p_{j_n}-1)\to 1$ as $n\to\infty$.

2. **Regularly varying sequences.** We call a sequence $\{\theta(n)\}$ of positive terms *regularly varying* if there is a sequence of positive terms $\{\alpha(n)\}$ satisfying

$$\theta(n) \sim K\alpha(n), \quad K \text{ a positive constant,} \tag{2.1a}$$

$$n(1 - \{\alpha(n-1)/\alpha(n)\}) \to \rho, \qquad \rho \text{ finite.} \tag{2.1b}$$

The number ρ will be called the index of regular variation. The case $\rho=0$ may be called slowly varying. It is first necessary to note that there exist regularly varying sequences $\{\theta(n)\}$ which themselves do not satisfy the condition (2.1b) with α replaced by θ (just as not all regularly varying functions can satisfy (1.5), although $R(x)\sim R_1(x)$ always). For example, if we take $\theta(n)=1+(-1)^n/n$, $n\geqq 2$, then $\theta(n)$ is regularly varying with index 0, since appropriate $\alpha(n)$ is given by $\alpha(n)=1$. However

$$\begin{aligned} n(1 - \{\theta(n-1)/\theta(n)\}) &\to -2, \qquad n\to\infty,\ n \text{ odd},\\ &\to 2, \qquad n\to\infty,\ n \text{ even}. \end{aligned}$$

We shall say that a sequence of positive terms $\{\theta(n)\}$, $n=0, 1, 2, \cdots$, is *imbeddable* in a regularly varying function R on $[0, \infty)$ if $R(n)=\theta(n)$, $n\geqq 0$.

LEMMA. *If $\{\theta(n)\}$, $n=0, 1, 2, \cdots$, is a regularly varying sequence of index ρ, then it has representation*

$$\theta(n) = n^\rho a(n)\exp\left\{\sum_{j=1}^{n}\frac{\varepsilon(j)}{j}\right\}, \qquad n\geqq 1, \tag{2.2}$$

where, as $n\to\infty$, $a(n)\to$positive limit, $\varepsilon(n)\to 0$.

PROOF. Since $\theta(n)\sim K\alpha(n)$, we may assume without loss of generality that $1-\alpha(m-1)/\alpha(m)\equiv\rho/m+\varepsilon(m)/m$, $m\geqq 1$, is less than unity in modulus for all $m\geqq 1$, by changing the first few terms of $\{\alpha(n)\}$. Since for $|x|<1$, $-\log(1-x)=\sum_{k=1}^{\infty} x^k/k$, we obtain

$$-\log\left\{\frac{\alpha(m-1)}{\alpha(m)}\right\} - \sum_{k=2}^{\infty}\frac{1}{k}\left\{1 - \frac{\alpha(m-1)}{\alpha(m)}\right\}^k = \frac{\rho}{m} + \frac{\varepsilon(m)}{m}.$$

Summing over m from 1 to n,

$$\log\alpha(n) - \log\alpha(0) - \sum_{m=1}^{n}\sum_{k=2}^{\infty}\frac{1}{k}\left\{1 - \frac{\alpha(m-1)}{\alpha(m)}\right\}^k = \rho\sum_{m=1}^{n}\frac{1}{m} + \sum_{m=1}^{n}\frac{\varepsilon(m)}{m}.$$

Now it is well known that $\sum_{m=1}^{n} m^{-1} - \log n = \gamma + o(1)$ as $n \to \infty$, where γ is a positive constant. Further since for each integer $k \geqq 2$, from (2.1b),

$$|\{1 - \alpha(m-1)/\alpha(m)\}^k| < (|\rho| + \delta_1)^k/m^k$$

for arbitrary fixed positive δ_1, and positive integer m sufficiently large (independently of k), we have the upper bound

$$= ((|\rho| + \delta_1)/m^{1/4})^k/m^{3k/4} \leqq \delta_2^k/m^{3/2}$$

for $k \geqq 2$ and with $0 < \delta_2 < 1$, for m large (independent of k). Thus we obtain that the series $\sum_{m=1}^{\infty} \sum_{k=2}^{\infty} k^{-1}\{1 - \alpha(m-1)/\alpha(m)\}^k$ is (absolutely) convergent. Hence it follows that

$$\alpha(n) = n^\rho a(n) \exp\left\{\sum_{j=1}^{n} \frac{\varepsilon(j)}{j}\right\}$$

where, as $n \to \infty$, $a(n) \to$ positive limit, $\varepsilon(n) \to 0$. Since $K(n) \equiv \theta(n)/\alpha(n) \to$ pos. const. by (2.1a), it follows that $\theta(n)$ has the same kind of representation, as required.

THEOREM. *A sequence of positive terms* $\{\theta(n)\}$, $n \geqq 0$, *is imbeddable in a regularly varying function R on* $[0, \infty)$ *if and only if the sequence is also regularly varying.*

PROOF. *Sufficiency.* If $\{\theta(n)\}$ is regularly varying with index ρ, we have representation (2.2) available for $\theta(n)$, $n \geqq 1$. Put (where $[u]$ denotes the integer part of u)

$$R(0) = \theta(0),$$
$$R(x) = x^\rho a([x]) \exp\left\{\int_0^x \frac{\varepsilon([t+1])}{[t+1]}\, dt\right\},$$

for $x > 0$, defining $a(0)$ by $a(0) = 1$, say. A glance at (1.4), or a direct verification using the definition of a regularly varying function, shows $R(x)$ is regularly varying with index ρ, and $R(n) = \theta(n)$, $n \geqq 0$.

Necessity. Conversely, if $R(x)$ is a regularly varying function on $[0, \infty)$ of index ρ then we have, for $x \geqq B \geqq 0$,

$$R(x) = x^\rho \exp\left\{\eta(x) + \int_B^x \frac{\varepsilon(t)}{t}\, dt\right\}$$

from (1.4), where $\varepsilon(t)$ may be taken as continuous for $x \geqq B$, $\varepsilon(t) \to 0$ as $t \to \infty$, $\eta(x) \to c$ $(|c| < \infty)$. For integer $n \geqq 0$, we have $\theta(n) = R(n)$, so for integer $n \geqq B$

$$\theta(n) = n^\rho \exp\left\{\eta(n) + \int_B^n \frac{\varepsilon(t)}{t}\, dt\right\},$$

since we are assuming $\theta(n)$ is imbeddable in $R(x)$.

To verify (2.1a) and (2.1b), put $\alpha(n) = n^\rho \exp\{\int_B^n \varepsilon(t)/t\, dt\}$ for all sufficiently large n; then $\theta(n) \sim K\alpha(n)$, K a positive constant; also

$$n\left(1 - \frac{\alpha(n-1)}{\alpha(n)}\right) = n\left(1 - (1 - 1/n)^\rho \exp\left\{-\int_{n-1}^n \frac{\varepsilon(t)}{t}\, dt\right\}\right) \to \rho$$

by using power series expansions, noting that $n \int_{n-1}^n \varepsilon(t)/t\, dt = n[\varepsilon(\xi_n)/\xi_n]$ where $n-1 < \xi_n < n$ by the mean value theorem ($\varepsilon(t)$ being continuous).

COROLLARY. *If a sequence of positive terms $\{\theta(n)\}$, $n \geqq 0$, is regularly varying with index ρ, then so is the positive function $\theta(x)$, $x \in [0, \infty)$, defined in terms of the sequence by $\theta(x) \equiv \theta([x])$, $x \geqq 0$.*

PROOF. Let $R(x)$, $x \in [0, \infty)$, be a regularly varying function of index ρ, in which $\{\theta(n)\}$ is imbeddable. Then, for $x > 0$,

$$\begin{aligned}\theta(x) &= \theta([x]) = R([x]) = R(x - \delta_x), \qquad \text{where } 0 \leqq \delta_x < 1;\\ &= R(x(1 - (\delta_x/x))), \quad \sim R(x), \quad \text{as } x \to \infty,\end{aligned}$$

by the uniform convergence property of regularly varying functions.

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