FOR HOW MANY EDGES IS A DIGRAPH ALMOST CERTAINLY HAMILTONIAN?

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ABSTRACT. $K_n(r)$ is the number of those Hamiltonian cycles in the complete digraph on n nodes, each of which has just r edges in common with a particular Hamiltonian cycle, and $B(n, r) = n!/\{r!(n-r)!\}$. We show that $K_n(r) = B(n, r)K_{n-r}(0)$ and deduce that $K_n(0) \sim (n-1)!e^{-1}$ for large n and that $K_n(r) \sim (n-1)!e^{-1}/r!$ if $r = o(n)$. An (n, q) digraph is one with n labelled nodes and q edges. From the result for $K_n(r)$, we deduce that, if $qn^{-3/2} \to \infty$ as $n \to \infty$, then almost all (n, q) digraphs are Hamiltonian. If $qn^{-3/2} \to c$ as $n \to \infty$, then the proportion of (n, q) digraphs which are non-Hamiltonian is at most $1 - \exp(-c^{-2})$ as $n \to \infty$.

An (n, q) digraph consists of n nodes and q directed edges. We consider only labelled digraphs, i.e. digraphs whose nodes are labelled. If A, B are two different nodes, they may be not joined or joined by the edge AB, or by the different edge BA or by both AB and BA. The greatest possible number of edges is $N = n^2 - n$ and the (n, N) digraph is called the complete digraph. We write $B(h, k) = h!/\{k!(h-k)!\}$, the usual binomial coefficient. Then obviously there are just $B(N, q)$ different (n, q) digraphs. We call a sequence of n edges

$$(1) \qquad A_1A_2, \; A_2A_3, \; A_3A_4, \; \cdots, \; A_{n-1}A_n, \; A_nA_1,$$

(where $A_1, \cdots, A_n$ are all the different n nodes of the graph) a *Hamiltonian cycle* (H.c.) and we say that a digraph which contains such a cycle is *Hamiltonian*. Our principal object is to find (so far as we can) for what q almost all (n, q) graphs are Hamiltonian.

We write $K_n(r)$ for the number of H.c. in the complete graph, each of which has just r edges in common with the specified H.c. in (1). The asymptotic approximation to $K_n(r)$ for large n and appropriate r is essential to our method, which is otherwise that used by O'Neil [2] to study the

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permanents of $(0, 1)$ matrices. In his case, the number $K_n(r)$ is replaced by D_n, Euler's rencontre number, whose asymptotic approximation, viz. $D_n = n!e^{-1} + O(1/n)$, was found by Euler [1]. I find here an exact value for $K_n(r)$, which is perhaps of some interest, but only the asymptotic approximation is needed for my main results.

If we apply the method used here to study Hamiltonian *graphs*, the number $K_n(r)$ is replaced by $H_n(r)$. I can find an asymptotic approximation to $H_n(r)$ for large n when $r^2 = o(n)$ and this is enough to prove my principal result about Hamiltonian graphs. But the study of $H_n(r)$ is more troublesome than that of $K_n(r)$, so I postpone this. Palasti [3] has calculated $H_n(n-2)$, $H_n(n-3)$ and $H_n(n-4)$ and found other results about $H_n(r)$. These illustrate the relative complexity of $H_n(r)$ compared with $K_n(r)$.

Obviously $K_n(n) = 1$, $K_n(n-1) = 0$ and $K_3(0) = 1$. We define $K_0(0) = 1$, $K_1(0) = 0$. We prove first that

$$(2) \qquad K_n(r) = B(n, r)K_{n-r}(0).$$

Clearly (2) is true for $r = n$ and $r = n-1$. We suppose $r \leq n-2$. We can choose the set of r edges out of the n edges in (1) in just $B(n, r)$ ways. Take any one such set. In the cycle (1) contract each of these r edges (or each consecutive run of them) and their end-nodes into a *special* node. We are left with an H.c. on $n-r$ nodes, of which r or less are special. In the complete digraph on $n-r$ nodes there are just $K_{n-r}(0)$ different H.c. which have no edges in common with our H.c. contracted from (1). If in each of these we expand each special node appropriately (i.e. by replacing it by the nodes and edge or edges from which it was contracted), we have an H.c. in the complete graph on n nodes which has just r edges in common with the H.c. in (1). There is no ambiguity about how a special node is replaced by the contracted edge or edges, since the direction of the new H.c. determines this in just one way. Thus the number of H.c. in the complete graph on n nodes which has just the prescribed set of r edges in common with the H.c. in (1) is $K_{n-r}(0)$. The result (2) follows.

There are just $(n-1)!$ different H.c. in the complete digraph on n nodes and so

$$(3) \qquad (n-1)! = \sum_{r=0}^{n} K_n(r).$$

If we use (2) and put $k_0 = 1$, $k_1 = 0$ and $k_n = K_n(0)/n!$ when $n \geq 2$, (3) becomes

$$1/n = \sum_{r=0}^{n} k_{n-r}/r!$$

where, as usual, $0!=1$. Hence, if $|X|<1$ and $W(X)=\sum_{n=0}^{\infty} k_n X^n$, we have

$$1 + \sum_{n=1}^{\infty} X^n/n = e^X W(X)$$

and so

$$W(X) = e^{-X}\left(1 + \sum_{n=1}^{\infty} (X^n/n)\right).$$

Hence, for $n \geq 1$,

$$k_n = \sum_{r=0}^{n-1} (-1)^r/\{r!\,(n-r)\} + (-1)^n/n!$$

and so

$$K_n(0) = n! \sum_{r=0}^{n-1} (-1)^r/\{r!\,(n-r)\} + (-1)^n.$$

Hence

(4) $$K_n(0) + K_{n-1}(0) = (n-1)! \sum_{r=0}^{n-1} (-1)^r/r! = D_{n-1},$$

where D_{n-1} is Euler's rencontre number [1]. By (3), $K_{n-1}(0) \leq (n-2)!$, and so, since $D_{n-1} \sim (n-1)!\,e^{-1}$, we have

(5) $$K_n(0) \sim (n-1)!\,e^{-1}.$$

Before using (5) for our main purpose, we remark that (4) gives us

$$K_n(0) = \sum_{s=1}^{n-3} (-1)^{s-1} D_{n-s} + (-1)^{n-1},$$

since $K_3(0)=1$.

We now denote by π any H.c. in the complete digraph on n nodes; π is associated with just one permutation of the nodes $A_2, A_3, \cdots, A_n$ in (1). For any particular digraph we write $E_\pi=1$, if the digraph contains π, and $E_\pi=0$, if not. We write $E=\sum_\pi E_\pi$, so that the digraph is or is not Hamiltonian according as $E>0$ or $E=0$. It is clear that π occurs in just $B(N-n, q-n)$ of the (n, q) digraphs. Hence, if $\sum'$ denotes summation over all (n, q) digraphs, we have

$$\sum{}' E = \sum_\pi \sum{}' E_\pi = (n-1)!\,B(N-n, q-n)$$

and so $M(E)$, the average value of E over all (n, q) digraphs, is

$$M(E) = (n-1)!\,B(N-n, q-n)/B(N, q).$$

Again $\sum' E^2 = \sum_\pi \sum' E_\pi^2 + \sum_{\pi \neq \pi'} \sum' E_\pi E_{\pi'}$. If π, π' have just r edges in common, they occur together in just $B(N-2n+r, q-2n+r)$ of the (n, q) digraphs and so $\sum' E_\pi E_{\pi'} = B(N-2n+r, q-2n+r)$. But there are

just $(n-1)!K_n(r)$ pairs π, π' (order relevant) which have just r edges in common and so

$$\sum{}' E^2 = (n-1)! \sum_{r=0}^{n} K_n(r)B(N-2n+r, q-2n+r).$$

For a non-Hamiltonian digraph, we have $E=0$ and so $(E-M(E))^2 = (M(E))^2$. Hence the number of non-Hamiltonian (n,q) digraphs is not greater than $\sum' \{E-M(E)\}^2/\{M(E)\}^2$. But $M\{E-M(E)\}^2 = M(E^2) - \{M(E)\}^2$ and so the proportion of (n,q) digraphs which are not Hamiltonian is not greater than $[M(E^2)/\{M(E)\}^2]-1$. Now

$$\frac{M(E^2)}{\{M(E)\}^2} = \frac{B(N,q) \sum' E^2}{(\sum' E)^2} = \sum_{r=0}^{n} \frac{\omega(r)K_n(r)}{(n-1)!},$$

where

$$\omega(r) = B(N,q)B(N-2n+r, q-2n+r)\{B(N-n, q-n)\}^{-2}$$
$$= \prod_{s=0}^{n-1} \left(\frac{N-s}{q-s}\right) \prod_{s=0}^{n-r-1} \left(\frac{q-n-s}{N-n-s}\right).$$

We now take $q=n^{3/2}\psi$, where $\psi=\psi(n)\to\infty$ as $n\to\infty$. Since the proportion of (n,q) digraphs which are Hamiltonian is nondecreasing as q increases we may without loss of generality suppose that $\psi < \log n$. We have

$$\log \omega(r) = r\log(N/q) + \sum{}'' (\log\{1-(s/N)\} - \log\{1-(s/q)\})$$
$$= r\log(N/q) + (q^{-1} - N^{-1})\sum{}'' s + o(1),$$

where

$$\sum{}'' s = \left(\sum_{s=0}^{n-1} - \sum_{s=n}^{2n-r-1}\right)s = -n^2 + 2nr - \tfrac{1}{2}r^2 + o(q).$$

Hence, for $r \leq n^{5/8}$ (say), we have

$$\log \omega(r) = r\rho - (n^2/q) + 1 + o(1),$$

where $\rho = \rho(n,q) = \log(N/q) + (2n/q)$, while, for $r > n^{5/8}$, we have

$$\log \omega(r) \leq r\rho - (n^2/q) + 1 + o(1).$$

Hence, if we put $m = 3[e^\rho]$, we see that $\sum_{r=m+1}^{n} e^{r\rho}/r! = o(1)$ and

$$M(E^2)\{M(E)\}^{-2} \sim \exp(-n^2 q^{-1})\left\{\sum_{r=0}^{m} (e^{r\rho}/r!) + o(1)\right\}$$
$$\sim \exp\{e^\rho - (n^2 q^{-1})\},$$

where $e^\rho - n^2 q^{-1} = (Ne^{2n/q} - n^2)/q = o(1)$. Hence $M(e^2)\{M(e)\}^{-2} \sim 1$ and so the proportion of non-Hamiltonian (n, q) digraphs is $o(1)$.

We have thus proved the following theorem.

THEOREM 1.   *If $qn^{-3/2} \to \infty$ as $n \to \infty$, then almost all labelled (n, q) digraphs are Hamiltonian.*

If we take our expansion of $\log \omega(r)$ to one further term, i.e. consider $\sum'' s^2$, we can prove the following theorem.

THEOREM 2.   *If $qn^{-4/3} \to \infty$ as $n \to \infty$ then $M(e^2)\{M(e)\}^{-2} \sim \exp(n^3 q^{-2})$.*

This shows that we cannot improve Theorem 1 by our method. But it is very unlikely that the result is best possible. A little calculation however enables us to prove the following result.

THEOREM 3.   *If $qn^{-3/2} \to c$ as $n \to \infty$, then the proportion of non-Hamiltonian labelled (n, q) digraphs is not more than $1 - e^{-c^{-2}}$ in the limit.*

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