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# ON INTEGRATED SCREENS

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ABSTRACT. Let $\mathscr{L}$ be a screen with support $\pi$ and let $\mathscr{F}$ denote the saturated formation of finite solvable groups which is locally induced by $\mathscr{L}$. For each prime $p$, let $\mathscr{M}(p)=\mathscr{L}(p)\cap\mathscr{F}$. Then $\mathscr{M}$ is an integrated screen which locally induces $\mathscr{F}$ and $\mathscr{M}\subseteq\mathscr{L}$. The purpose of this note is to prove the following theorems. THEOREM 1. *Assume that for each finite solvable group G the $\mathscr{L}$-izers of G satisfy the strict cover-avoidance property. Then $\mathscr{L}$ is an integrated screen; that is $\mathscr{L}(p)\subseteq\mathscr{F}$ for each prime p.* THEOREM 2. *Assume that for each group G an $\mathscr{M}$-izer of an $\mathscr{L}$-izer of G is an $\mathscr{M}$-izer of G. Then $\mathscr{L}(p)=\mathscr{M}(p)$ for each prime p.*

1. All groups considered are finite and solvable. The definitions and notation are standard and can be found in Carter and Hawkes [1], Seitz and Wright [3], and Wright [4].

Let $\mathscr{L}$ be a screen with support $\pi$ and let $\mathscr{F}$ denote the saturated formation locally induced by $\mathscr{L}$. Let $D$ be an $\mathscr{L}$-izer of the finite solvable group $G$. By Theorem 3 of [4], $D$ covers each $\mathscr{L}$-central chief factor of $G$ and avoids each $\mathscr{L}$-eccentric chief factor of $G$. In particular, $D$ has the cover-avoidance (c.a.) property. $D$ is said to satisfy the strict cover-avoidance (s.c.a.) property if whenever $D$ covers the chief factor $H/K$ of $G$, then $H\cap D/K\cap D$ is a chief factor of $D$. Example 2 of [1, p. 189] shows that an $\mathscr{L}$-izer need not satisfy the s.c.a. property. However, a slight generalization of Theorem 4.1 of [1] yields: If $\mathscr{L}$ is an integrated screen; that is $\mathscr{L}(p)\subseteq\mathscr{F}$ for each $p\in\pi$; then $D$ satisfies the s.c.a. property. In this paper we prove the converse, namely;

THEOREM 1. *Let $\mathscr{L}$ be a screen with support $\pi$. Assume that for each group G the $\mathscr{L}$-izers of G satisfy the s.c.a. property in G. Then $\mathscr{L}$ is an integrated screen.*

Let $\mathscr{L}$ and $\mathscr{M}$ be screens. Write $\mathscr{M}\subseteq\mathscr{L}$ if $\mathscr{M}(p)\subseteq\mathscr{L}(p)$ for each prime $p$. Then "$\subseteq$" is a partial order on the set of screens. In Theorem 5 of [4] Wright proves: If $\mathscr{L}$ is an integrated screen and $\mathscr{M}\subseteq\mathscr{L}$, then an $\mathscr{M}$-izer of an $\mathscr{L}$-izer of the group $G$ is an $\mathscr{M}$-izer of $G$. The converse of Wright's theorem is contained in the following theorem.

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THEOREM 2. *Let $\mathscr{L}$ be a screen with support $\pi$ and let $\mathscr{F}$ denote the saturated formation locally induced by $\mathscr{L}$. Let $\mathscr{M}$ denote the screen given by $\mathscr{M}(p)=\mathscr{L}(p)\cap\mathscr{F}$ for each prime p. Then $\mathscr{M}\subseteq\mathscr{L}$ and $\mathscr{M}$ is an integrated screen which locally induces $\mathscr{F}$. Assume that for each group G an $\mathscr{M}$-izer of an $\mathscr{L}$-izer of G is an $\mathscr{M}$-izer of G. Then $\mathscr{M}(p)=\mathscr{L}(p)$ for each prime p.*

2. **Proof of Theorem 1.** Assume that there is a prime $p\in\pi$ such that $\mathscr{L}(p)$ is not contained in the saturated formation $\mathscr{F}$ locally induced by $\mathscr{L}$. Let $G$ be a group of minimal order such that $G\in\mathscr{L}(p)$ but $G\notin\mathscr{F}$. Hence, $G$ contains a unique minimal normal subgroup $A=C_G(A)$ and $A$ is complemented by a maximal subgroup $F$ of $G$. Let $|A|$ be a power of the prime $q$. By Theorem 8 of **[4]** $F$ is an $\mathscr{F}$-projector of $G$ as well as an $\mathscr{L}$-izer of $G$. By Theorem 3(A) of **[4]**, $p\neq q$. From Hilfssatz 1.2 of **[2]** there exists a faithful, irreducible $GF(p)[G]$-module $V$ such that $V_F$ has an irreducible factor module $V/V_0$ on which $F$ acts trivially. Let $H$ be the semidirect product of $V$ with $G$. Then $V$ is a minimal normal subgroup of $H$, $H/C_H(V)\in\mathscr{L}(p)$ and so $V$ is $\mathscr{L}$-central in $H$. Let $D$ be an $\mathscr{L}$-izer of $H$. Then $D$ covers $V$ by Theorem 3(A) of **[4]**, hence we can assume that $D/V=FV/V$. Since $D$ has the s.c.a. property, $V_D$ is irreducible and so $V_0$ is the identity module. Hence, $F$ acts trivially on $V$, a contradiction. Thus $\mathscr{L}(r)\subseteq\mathscr{F}$ for each $r\in\pi$ and the theorem follows.

3. **Proof of Theorem 2.** Assume that there is a prime $q\in\pi$ such that $\mathscr{L}(q)$ is not contained in $\mathscr{F}$. Let $G$ be a group of minimal order such that $G\in\mathscr{L}(q)$, but $G\notin\mathscr{F}$. Then $G$ contains a unique minimal normal subgroup $A=C_G(A)$ and $A$ is complemented by a maximal subgroup $F$ of $G$. Let $|A|$ be a power of $p$, then $p\neq q$. By Theorem 8 of **[4]** $F$ is an $\mathscr{L}$-izer of $G$ as well as an $\mathscr{F}$-projector.

By Hilfssatz 1.3 of **[2]** there exists a faithful, irreducible $GF(q)[G]$-module $V$. Let $H$ be the semidirect product of $V$ with $G$. Then $V$ is the unique minimal subgroup of $H$, $C_H(V)=V$, and $H/V\in\mathscr{L}(q)$. Let $D$ be an $\mathscr{L}$-izer of $H$. By Theorem 3 of **[4]** $D$ covers $V$ and $D/V$ is an $\mathscr{L}$-izer of $H/V$, hence we can assume that $D=FV$. Also note that $D/V\in\mathscr{L}(q)$. Let $H/K$ be a chief factor of $D$ lying below $V$. Then $H/K$ is a $q$-chief factor of $D$ centralized by $V$, hence $D/C_D(H/K)\in\mathscr{L}(q)$. Hence, each chief factor of $D$ is $\mathscr{L}$-central and so $D\in\mathscr{F}$.

Let $E$ be an $\mathscr{M}$-izer of $D$. Then $E=D$ and $D$ is an $\mathscr{M}$-izer of $H$. Because of Theorem 3(A) of **[4]**, $V$ is $\mathscr{M}$-central in $H$, hence

$$G\cong H/V\in\mathscr{M}(p)$$

which is a contradiction. This completes the proof.

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