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HIGHER DERIVATIONS ON FINITELY GENERATED INTEGRAL DOMAINS

W. C. BROWN

ABSTRACT. In this paper, we prove the following theorem: Let $A=k[x_1, \cdots, x_t]$ be a finitely generated integral domain over a field k of characteristic zero. Then A regular, i.e. the local ring A_q is regular for all primes $q \subseteq A$, is equivalent to the following two conditions: (1) No nonminimal prime of A is differential, and (2) $\mathrm{der}^n(A/k)=\mathrm{Der}^n(A/k)$ for all n. Here $\mathrm{Der}^n(A/k)$ denotes the A-module of all nth order derivations of A into A which are zero or k, and $\mathrm{der}^n(A/k)$ denotes the A-submodule of $\mathrm{Der}^n(A/k)$ generated by composites $\delta_1 \circ \cdots \circ \delta_j$ $(1 \leqq j \leqq n)$ of first order derivations δ_i.

Introduction. Throughout this paper we assume k is a field of characteristic zero. Let $A=k[x_1, \cdots, x_t]$ be a finitely generated integral domain over k. We shall let $\mathrm{Der}^n(A/k)$ denote the A-module of nth order derivations (see [6]) of A to itself which vanish on k. It follows from Proposition 4 and Corollary 6.1 of [6] that any composite $\delta_1 \circ \cdots \circ \delta_j$ $(1 \leqq j \leqq n)$ of j-derivations $\delta_i \in \mathrm{Der}^1(A/k)$ is an nth order derivation in $\mathrm{Der}^n(A/k)$. The A-submodule of $\mathrm{Der}^n(A/k)$ spanned by all such composites will be denoted by $\mathrm{der}^n(A/k)$.

In general, one would like to know under what conditions is $\mathrm{der}^n(A/k)=\mathrm{Der}^n(A/k)$ for all n. Recently K. Mount and O. E. Villamayor in [4] obtained a result for domains of dimension one over k. They proved the following result: Let A be the coordinate ring of an irreducible algebraic curve over k. Let p be a nonzero prime ideal of A. Then A_p is a regular local ring if and only if $\mathrm{der}^n(A_p/k)=\mathrm{Der}^n(A_p/k)$ for all n. Let us say a finitely generated domain $A=k[x_1, \cdots, x_t]$ over k is regular if A_q is a regular local ring for all prime ideals $q \subseteq A$. Since $\mathrm{der}^n(A/k) \otimes_A A_q \cong \mathrm{der}^n(A_q/k)$ and $\mathrm{Der}^n(A/k) \otimes_A A_q \cong \mathrm{Der}^n(A_q/k)$, the result of Mount and Villamayor can be restated as follows: Let $A=k[x_1, \cdots, x_t]$ be a finitely generated integral domain over a field k of characteristic zero. Suppose A has dimension one over k, i.e., the quotient field of A has transcendence degree one over k. Then A is regular if and only if $\mathrm{der}^n(A/k)=\mathrm{Der}^n(A/k)$ for all n.

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We shall say that a prime ideal $p \subseteq A$ is differential if $\delta(p) \subseteq p$ for all $\delta \in \mathrm{Der}^1(A/k)$. In this paper, we shall prove the following result: Let $A = k[x_1, \cdots, x_t]$ be a finitely generated integral domain over a field k of characteristic zero. Then A is regular if and only if (1) no nonminimal prime ideal $q \subseteq A$ is differential, and (2) $\mathrm{der}^n(A/k) = \mathrm{Der}^n(A/k)$ for all n.

Main results. We begin with the following fundamental theorem.

THEOREM 1. *Let $A = k[x_1, \cdots, x_t]$ be a finitely generated integral domain over a field k of characteristic zero. Then if* (1) *no nonminimal prime ideal $q \subseteq A$ is differential and* (2) $\mathrm{der}^n(A/k) = \mathrm{Der}^n(A/k)$ *for all n, then A is integrally closed.*

PROOF. The proof is by contradiction. Let Q denote the quotient field of A, and let $\bar{A}$ denote the integral closure of A in Q. We assume $A \neq \bar{A}$. Thus, the conductor $C = A : \bar{A}$ is a proper ideal in A, i.e., $0 < C < A$. Let p be an associated prime of C. It follows from [7, Corollary, p. 169] that C is a differential ideal in A. Since p is an associated prime of C, p is a differential prime ideal in A [8, Theorem 1]. Thus by hypothesis, p is a minimal prime ideal of A.

If R is any k-algebra, we shall denote by $D^1(R/k)$ the R-module of first order differentials of R over k (see [5]).

Now consider the local ring A_p. The integral closure $\bar{A}_p$ of A_p in Q is a finitely generated A_p-module and hence a semilocal ring. Let $\{\bar{p}_1, \cdots, \bar{p}_m\}$ denote the maximal ideals of $\bar{A}_p$. Since $(\bar{A}_p)_{\bar{p}_i} = V_i$ is a discrete rank one valuation ring, [5, Theorem 3′] implies that $D^1(V_i/k)$ is a free V_i-module of rank r. Here r is the transcendence degree of A over k. It follows that $D^1(\bar{A}_p/k)$ is a projective $\bar{A}_p$-module of rank r. Thus, since $\bar{A}_p$ is semilocal, $D^1(\bar{A}_p/k)$ is a free $\bar{A}_p$-module of rank r.

Let $d: A_p \to D^1(A_p/k)$ and $\bar{d}: \bar{A}_p \to D^1(\bar{A}_p/k)$ denote the canonical k-derivations of A_p and $\bar{A}_p$ into $D^1(A_p/k)$ and $D^1(\bar{A}_p/k)$ respectively. If $K(A/p)$ denotes the quotient field of A/p, then $K(A/p)$ has transcendence degree $r-1$ over k. Thus, there exist elements $\alpha_1, \cdots, \alpha_{r-1} \in A - p$ such that $K(A/p)$ is a separable algebraic extension of $k(\bar{\alpha}_1, \cdots, \bar{\alpha}_{r-1})$. Here $\bar{\alpha}_i$ of course denotes the image of α_i in A/p. We note that $F = k(\alpha_1, \cdots, \alpha_{r-1})$ is a field contained in A_p.

By [9, Theorem 18, p. 45], there exists a $\beta \in \bigcap_{i=1}^m \bar{p}_i$ such that β generates the maximal ideal of each V_i. From the proof of [5, Theorem 3′], we have the following short exact sequence of V_i-modules:

$$0 \to V_i \otimes_F D^1(F/k) \to D^1(V_i/k) \to D^1(V_i/F) \to 0. \tag{1}$$

It easily follows from (1) that $\{d_i(\beta), d_i(\alpha_1), \cdots, d_i(\alpha_{r-1})\}$ is a free V_i-basis of $D^1(V_i/k)$. Here $d_i: V_i \to D^1(V_i/k)$ is the canonical derivation. An

application of Nakayama's lemma shows that $\{\bar{d}(\beta), \bar{d}(\alpha_1), \cdots, \bar{d}(\alpha_{r-1})\}$ is a free basis of $D^1(\bar{A}_p/k)$.

Let $\{\Psi_0, \Psi_1, \cdots, \Psi_{r-1}\} \subset \mathrm{Hom}_{\bar{A}_p}(D^1(\bar{A}_p/k), \bar{A}_p)$ be a dual basis to $\{\bar{d}(\beta), \bar{d}(\alpha_1), \cdots, \bar{d}(\alpha_{r-1})\} \subset D^1(\bar{A}_p/k)$. Then setting $\delta_i = \Psi_i \circ \bar{d}$, we get r k-derivations of $\bar{A}_p$ such that $\delta_0(\beta) = \delta_i(\alpha_i) = 1$, $\delta_0(\alpha_i) = \delta_i(\beta) = 0$ and $\delta_i(\alpha_j) = 0$ if $i \neq j$. Since $\mathrm{Hom}_{\bar{A}_p}(D^1(\bar{A}_p/k), \bar{A}_p) \cong \mathrm{Der}^1(\bar{A}_p/k)$, we see that $\{\delta_0, \delta_1, \cdots, \delta_{r-1}\}$ is a free $\bar{A}_p$-module basis for $\mathrm{Der}^1(\bar{A}_p/k)$. Now if $\varphi \in \mathrm{Der}^1(A_p/k)$, then by [7, Theorem] φ extends uniquely to $\hat{\varphi} \in \mathrm{Der}^1(\bar{A}_p/k)$. Thus, there exist $a_0, \cdots, a_{r-1} \in \bar{A}_p$ such that

$$\hat{\varphi} = a_0\delta_0 + a_1\delta_1 + \cdots + a_{r-1}\delta_{r-1}.$$

We wish to characterize a_0.

Now $\bar{d}$ when restricted to A_p gives a k-derivation of A_p into $D^1(\bar{A}_p/k)$. Hence, by the universal mapping property of $D^1(A_p/k)$, there exists a unique A_p-module homomorphism $\sigma: D^1(A_p/k) \to D^1(\bar{A}_p/k)$ such that $\sigma \circ d = \bar{d}|_{A_p}$. Since $d(\alpha_i) \in D^1(A_p/k)$, $\bar{d}(\alpha_i) \in \mathrm{Im}\,\sigma$. Thus, we can write

$$\mathrm{Im}\,\sigma = M\bar{d}(\beta) \oplus A_p\bar{d}(\alpha_1) \oplus \cdots \oplus A_p\bar{d}(\alpha_{r-1})$$

for some A_p-submodule $M \subset \bar{A}_p$. If $T = \ker \sigma$, then

$$D^1(A_p/k) \otimes_{A_p} Q \cong D^1(Q/k) \cong D^1(\bar{A}_p/k) \otimes_{A_p} \bar{A}_p$$

implies that T is a torsion submodule of $D^1(A_p/k)$. Thus, we have

$$\begin{aligned} \mathrm{Hom}_{A_p}(\mathrm{Im}\,\sigma, A_p) &\cong \mathrm{Hom}_{A_p}(D^1(A_p/k)/T, A_p) \\ &\cong \mathrm{Hom}_{A_p}(D^1(A_p/k), A_p) \cong \mathrm{Der}^1(A_p/k). \end{aligned} \tag{2}$$

Now, if $M = (0)$, then $\mathrm{Im}\,\sigma$ is a free A_p-module. Hence by (2), $\mathrm{Der}^1(A_p/k)$ is a free A_p-module. But then [3, Theorem 1] implies that A_p is normal which is a contradiction. Thus, M is a nonzero submodule of $\bar{A}_p$.

We next note that if $\varphi \in \mathrm{Der}^1(A_p/k)$, and we write $\hat{\varphi} = a_0\delta_0 + a_1\delta_1 + \cdots + a_{r-1}\delta_{r-1}$, then $a_0M \leqq A_p$. If $a_0M = A_p$, then M is a free A_p-module. Consequently, $\mathrm{Im}\,\sigma$ is free, and we again reach a contradiction. Thus, $a_0M \leqq pA_p$, the maximal ideal of A_p.

Now let $\{z_1, \cdots, z_l\}$ be a minimal basis for the maximal ideal pA_p. By [5, (G)] the following sequence of A_p/pA_p-modules is exact:

$$0 \to pA_p/p^2A_p \to (A_p/pA_p) \otimes_{A_p} D^1(A_p/k) \to D^1((A_p/pA_p)/k) \to 0. \tag{3}$$

Since $A_p/pA_p \cong K(A/p)$ is a separable algebraic extension of F, (3) implies that $D^1(A_p/k)$ is generated as an A_p-module by $\{d(z_1), \cdots, d(z_l), d(\alpha_1), \cdots, d(\alpha_{r-1})\}$. Thus, $\{\delta_0(z_1), \cdots, \delta_0(z_l)\}$ generate M.

The argument from this point on is essentially that found in [4] applied to the $\mathrm{Der}^1(A_p/F)$-components of $\mathrm{Der}^1(A_p/k)$. If v_i denotes the valuation on V_i, then $v_i(\delta_0(z_j))=v_i(z_j)-1$ for all $j=1,\cdots,l$ and $i=1,\cdots,m$. Thus if $\varphi\in\mathrm{Der}^1(A_p/k)$ and $\hat{\varphi}=a_0\delta_0+\cdots+a_{r-1}\delta_{r-1}$, then $a_0\in\bigcap_{i=1}^m \bar{p}_i$. If $I=\{a_0\in\bar{A}_p \mid a_0\delta_0+\cdots+a_{r-1}\delta_{r-1}\in\mathrm{Der}^1(A_p/k)$ for some $a_i\in\bar{A}_p\}$, then $IV_1=(\beta^f)$ for some integer $f\geqq 1$. Set $C\cdot V_1=(\beta^c)$ and find $y\in C$ such that $v_1(y)=c$. Choose an integer N sufficiently large so that $fN>c$. Then $y\delta_0^N\in\mathrm{Der}^N(A_p/k)$ but $y\delta_0^N\notin\mathrm{der}^N(A_p/k)$. Thus hypothesis (2) in the theorem is contradicted and the proof is complete. □

We can now prove the main result of this paper.

THEOREM 2. *Let $A=k[x_1,\cdots,x_t]$ be a finitely generated integral domain over a field k of characteristic zero. Then A is regular if and only if* (1) *no nonminimal prime ideal $q\subseteq A$ is differential, and* (2) $\mathrm{der}^n(A/k)=\mathrm{Der}^n(A/k)$ *for all n.*

PROOF. If A is regular, then condition (1) follows from [8, Theorem 3]. Condition (2) follows from [2, Theorem 16.11, 2].

So conversely suppose A satisfies conditions (1) and (2). By Theorem 1, A is an integrally closed domain. Suppose q is a prime ideal of A of height one. Then A_q is a discrete rank one valuation ring and hence a regular local ring. Let us assume that A_q is a regular local ring for all primes q having height less than k. Here $1\leqq k<r$ the transcendence degree of A over k. Now suppose q is a prime ideal of height $k+1$ in A. Then by the induction hypothesis every proper localization of A_q is a regular local ring. If we assume A_q itself is not regular, then it follows from [8, Theorem 5] that qA_q is a differential ideal in A_q. But, q is a nonminimal prime of A and hence by hypothesis is not differential in A. Thus qA_q is not differential in A_q. Consequently, A_q must be regular and the proof is complete. □

REFERENCES

1. N. Bourbaki, *Eléments de mathématique*. Fasc. XXVII. *Algèbre commutative*. Chap. 1: *Modules plats*. Chap. 2: *Localisation*, Actualités Sci. Indust., no. 1290, Hermann, Paris, 1961. MR **36** #146.

2. A. Grothendieck, *Eléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. Nos. **4–32** (1960–67). MR **29** #1210; **30** #3885; **33** #7330; **36** #177a,b,c; **36** #178; **39** #220.

3. J. Lipman, *Free derivation modules on algebraic varieties*, Amer. J. Math. **87** (1965), 874–898. MR **32** #4130.

4. K. Mount and O. E. Villamayor, *On a question of Y. Nakai*, Osaka J. Math. (to appear).

5. Y. Nakai, *On the theory of differentials in commutative rings*, J. Math. Soc. Japan **13** (1961), 63–84. MR **23** #A2437.

6. Y. Nakai, *Higher order derivations*. I, Osaka J. Math. 7 (1970), 1–27. MR **41** #8404.

7. A. Seidenberg, *Derivations and integral closure*, Pacific J. Math. **16** (1966), 167–173. MR **32** #5686.

8. ———, *Differential ideals in rings of finitely generated type*, Amer. J. Math. **89** (1967), 22–42. MR **35** #2902.

9. O. Zariski and P. Samuel, *Commutative algebra*. II, University Series in Higher Math., Van Nostrand, Princeton, N.J., 1960. MR **22** #11006.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823