PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 2, February 1974

FACTORABLE BOUNDED OPERATORS AND SCHWARTZ SPACES

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ABSTRACT. A necessary condition for factoring continuous linear maps with domain c_0 or l_∞ through a class of spaces which include the l_p spaces (in fact, include the $\mathcal{L}_p$ spaces) for $2 \leqq p < \infty$ and a weaker result for l_1 are obtained. As an application, examples of Schwartz spaces are constructed and used to answer questions of Diestel, Morris and Saxon; in particular it is shown that there are Schwartz spaces which cannot be embedded in a product of l_p spaces, $1 < p < \infty$.

For $2 \leqq p < \infty$, we show the following: If S is a continuous linear map from c_0 or l_∞ into a normed space, then S is factorable through a space satisfying Clarkson's "parallelogram" laws for L_p spaces ([**1**] or see (1) below) only if a sequence, which is simply constructed from S, belongs to l_p. A similar result is obtained for diagonal maps on l_1. Examples are constructed to show that for no p, $1 < p < \infty$, does the variety $v(l_p)$ (defined below) contain the variety of all Schwartz spaces [**5**, p. 275]. In particular this shows that the theorem of Grothendieck [**4**] that any nuclear space can be topologically embedded in a product of l_p spaces, $1 \leqq p \leqq \infty$, (and Saxon's generalization [**9**]) cannot be extended to Schwartz spaces. Furthermore, an example borrowed from Pietsch [**8**] shows that the variety of nuclear spaces is properly contained in $\bigcap v(l_p)$, $1 \leqq p \leqq \infty$. These examples answer or partially answer questions raised by Diestel, Morris and Saxon in [**3**] or by Diestel and Morris in [**2**].

By a *map* we shall mean a continuous linear function. To say a map $S: X \to Y$ can be factored through Z means that there exist maps $T: X \to Z$ and $J: Z \to Y$ such that $S = JT$. Let e_n be the sequence with one in the nth place and zero otherwise. A *variety* (see [**2**] or [**3**]) is a collection of locally convex topological vector spaces closed with respect to taking subspaces, quotients by closed subspaces, products and isomorphic images. If B is a locally convex topological vector space, we let $v(B)$ be the

Received by the editors March 26, 1973 and, in revised form, April 30, 1973.

AMS (MOS) subject classifications (1970). Primary 46E30, 46A15; Secondary 47A99, 47B99.

Key words and phrases. Factorable maps, l_p-spaces, Schwartz spaces, nuclear spaces, varieties of topological vector spaces.

smallest variety containing B. For $2 \leqq p < \infty$, it is convenient to define a Q_p space to be any normed space X which has an equivalent norm $\|\cdot\|$ satisfying

$$(1) \qquad \forall x, y \in X, \qquad \|x+y\|^p + \|x-y\|^p \geqq 2(\|x\|^p + \|y\|^p).$$

Clarkson, in [1, Theorem 2, p. 400], shows that the usual norms of L_p and l_p satisfy (1) for $2 \leqq p < \infty$. It is easy to show that the property of being a Q_p space is preserved under taking subspaces, quotients by closed subspaces, and finite products. Thus any $\mathscr{L}_p$ space (see [6] or [7]) is a Q_p space for $2 \leqq p < \infty$ and any normed space in $\nu(l_p)$ [3, Theorem 4.1, p. 217] is a Q_p space for $2 \leqq p < \infty$. It is also fairly easy to show that for $1 < q \leqq 2$, the dual of any normed space in $\nu(l_q)$ is a Q_p space, where $p^{-1} + q^{-1} = 1$.

The following lemma singles out a result needed for the theorem; it also shows how Clarkson's "parallelogram" laws figure into the main result.

Lemma. *Let $2 \leqq p < \infty$, X a Q_p space, $T: c_0 \to X$ a map and (λ_n) a sequence such that $(\lambda_n^{-1}) \notin l_p$; then there exists a subsequence (n') of (n) such that $|\lambda_{n'}|\, \|Te_{n'}\| \to 0$ as $n' \to \infty$.*

Proof. Suppose not; then there exists an $\varepsilon > 0$ and an integer M, such that $n \geqq M$ implies $|\lambda_n|\, \|Te_n\| \geqq \varepsilon$. We show by induction that there is some choice of signs so that

$$(2) \qquad \|T(e_M \pm \cdots \pm e_{M+k})\|^p \geqq \varepsilon^p \sum_{j=0}^{k} |\lambda_{M+j}^{-1}|^p.$$

By assumption (2) is true for $k=0$. Suppose that (2) is true for $k=n$ with choice of signs $x_n = e_M \pm e_{M+1} \pm \cdots \pm e_{M+n}$. Now by (1) and the induction hypothesis we have

$$\begin{aligned} \|T(x_n + e_{M+n+1})\|^p &+ \|T(x_n - e_{M+n+1})\|^p \\ &\geqq 2(\|Tx_n\|^p + \|Te_{M+n+1}\|^p) \\ &\geqq 2\left(\varepsilon^p \sum_{j=0}^{n} |\lambda_{M+j}^{-1}|^p + \varepsilon^p |\lambda_{M+n+1}^{-1}|^p\right); \end{aligned}$$

so (2) is established. But this is impossible since for all k,

$$\|e_M \pm e_{M+1} \pm \cdots \pm e_{M+k}\| = 1, \qquad \sum |\lambda_{M+j}|^p$$

diverges and thus (2) implies that T is unbounded.

Theorem. *A necessary condition for the map S from c_0 or l_∞ into a normed space Y to be factorable through a Q_p space, $2 \leqq p < \infty$, is that the sequence $(\|Se_n\|)$ belong to l_p.*

PROOF. The theorem for c_0 clearly implies the theorem for l_∞. Let $2 \leqq p < \infty$ and suppose $S: c_0 \to Y$ can be factored through the Q_p space X by the maps $T: c_0 \to X$ and $J: X \to Y$, but that $(\|Se_n\|) \notin l_p$. Let $V: c_0 \to c_0$ be the map with $Ve_n = e_{\pi(n)}$ where the one-one function π from N *into* N (N the positive integers) is such that, for all n, $\|SVe_n\| \neq 0$ and $(\|SVe_n\|) \notin l_p$. By assumption, SV can be factored through X by the maps TV and J, that is, $SV = JTV$. The lemma implies there exists a subsequence (n') of (n) such that $\|SVe_{n'}\|^{-1} \|TVe_{n'}\| \to 0$ as $n' \to \infty$. But by continuity of J we have as $n' \to \infty$:

$$1 = \|SVe_{n'}\|^{-1} \, \|SVe_{n'}\| = \|J(\|SVe_{n'}\|^{-1} \, TVe_{n'})\| \to 0.$$

This contradiction completes the proof.

A *diagonal map* on a space Λ, of sequences, is a map $T_\lambda: \Lambda \to \Lambda$, where λ is a sequence (λ_n) and $T_\lambda(\mu_n) = (\lambda_n \mu_n)$.

COROLLARY. *A necessary condition for the diagonal map $T_\lambda: l_1 \to l_1$ to be factorable through a space X, whose dual is a Q_p space $(2 \leqq p < \infty)$ is that $\lambda = (\lambda_n) \in l_p$.*

PROOF. If $T_\lambda: l_1 \to l_1$ can be factored through X, then $T_\lambda^*: l_\infty \to l_\infty$ can be factored through X^*. The proof now follows from the theorem and the fact that the adjoint of T_λ, T_λ^* is the diagonal map $T_\lambda: l_\infty \to l_\infty$.

In [2] and [3] the following questions were raised:

(i) Is $\bigcap \nu(B)$, $B \in \mathscr{B}$ (where $\mathscr{B} = \{$all infinite dimensional Banach spaces$\}$), equal to S, the variety of all Schwartz spaces; N, the variety of all nuclear spaces, or neither?

(ii) Does $\nu(l_p)$, $1 < p < \infty$, contain S?

The following examples show that for no p, $1 < p < \infty$, is (ii) true and thus $\bigcap \nu(B)$, $B \in \mathscr{B}$, is properly contained in S [2]. Furthermore we show $\bigcap \nu(l_p)$, $1 \leqq p \leqq \infty$, properly contains N.

EXAMPLES. Let $\lambda = (\lambda_n)$ be any sequence converging to zero such that, for $k \geqq 1$, $\sum |\lambda_n|^k = \infty$, and let T_λ be the diagonal map. Let Λ_0 (respectively, Λ_1, Λ_2, Λ_∞) be the projective limit of the sequence:

$$\cdots \xrightarrow{T_\lambda} E \xrightarrow{T_\lambda} E \xrightarrow{T_\lambda} E,$$

where $E = c_0$ (respectively, $E = l_1$, $E = l_2$, $E = l_\infty$). Each of Λ_i, $i = 0, 1, 2, \infty$ is a Schwartz-Fréchet space [5, Proposition 9, p. 282], that is not nuclear. Λ_0 and Λ_∞ (respectively Λ_1) do not belong to the variety $\nu(l_p)$ for $2 \leqq p < \infty$ (respectively $1 < p \leqq 2$); in particular, for fixed p, it is not possible to topologically embed either in a product of l_p spaces $2 \leqq p < \infty$ (respectively $1 < p \leqq 2$). The above follows as the assertions are equivalent to the impossibility of factoring any finite number of iterates of T_λ through spaces X

covered by the theorem or the corollary (see the discussion preceding the lemma).

The space Λ_2 is an example of Pietsch [**8**, Satz 8, p. 122], where he shows that Λ_2 is a subspace of a product of l_p spaces for any p, $1<p<\infty$. Hence $\Lambda_2 \in \nu(l_p)$ for all p, $1\leqq p\leqq\infty$ (the cases $p=1$ and $p=\infty$ are true for any Schwartz space [**3**]).

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