A SHORT PROOF OF FAN'S FIXED POINT THEOREM

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ABSTRACT.   Fan's fixed point theorem for multivalued functions
in locally convex spaces is proved by means of Brouwer's fixed point
theorem and the concept of a partition of unity.

Let X and Y be topological spaces, and let 2^Y denote the set of subsets of Y. A function $T:X\to 2^Y$ (called a multivalued function of X into Y) is *upper semicontinuous* if for each $x_0 \in X$ and each neighborhood W of $T(x_0)$ in Y, there exists a neighborhood V of x_0 in X such that $x \in V$ implies $T(x) \subset W$. We shall give a short proof of Fan's fixed point theorem for multivalued functions, modifying a method of Browder [1].

THEOREM (K. FAN [2]).   *Let K be a nonempty compact convex subset of a Hausdorff locally convex topological vector space E, and let $T:K\to 2^K$ be an upper semicontinuous function such that $T(x)$ is nonempty closed and convex for each $x \in K$. Then there exists $x_0 \in K$ with $x_0 \in T(x_0)$.*

PROOF.   Let $\{U_i : i \in I\}$ denote a neighborhood base at 0 in E consisting of open convex circled sets. For each $i \in I$ there exists a finite set $\{x_{ij} : j \in J(i)\} \subset K$ with $K \subset \bigcup_{j \in J(i)} (x_{ij} + U_i)$. There exists a continuous partition of unity subordinate to this covering, i.e. for $j \in J(i)$ there are continuous functions $\alpha_{ij}:K\to R$ with $\alpha_{ij}(x) \geqq 0$ for $x \in K$, $\alpha_{ij}(x)=0$ for $x \notin x_{ij}+U_i$, and $\sum_{j \in J(i)} \alpha_{ij}(x)=1$ for $x \in K$. Choose $y_{ij} \in T(x_{ij})$ arbitrarily, and define the function $f_i:K\to E$ by $f_i(x)=\sum_{j \in J(i)} \alpha_{ij}(x)y_{ij}$. The convex hull C_i of $\{y_{ij} : j \in J(i)\}$ is homeomorphic to a Euclidean ball, with $C_i \subset K$ and $f_i(C_i) \subset C_i$. By Brouwer's fixed point theorem we may choose $x_i \in C_i$ such that $f_i(x_i)=x_i$.

The neighborhood base $\{U_i : i \in I\}$ is directed by $\supset$. Let $x_0 \in K$ be a cluster point of the corresponding net $\{x_i : i \in I\}$ in K, and suppose $x_0 \notin T(x_0)$. By separation there is a closed convex neighborhood W of $T(x_0)$ with $x_0 \notin W$. Since T is upper semicontinuous there exists a neighborhood V of x_0 with $V \cap W = \varnothing$ such that $x \in V \cap K$ implies $T(x) \subset W$. Choose $m \in I$ with $U_m + U_m \subset V - x_0$. There exists an $i \in I$ with $U_m \supset U_i$ such that $x_i \in x_0 + U_m$, and then $x_i + U_i \subset V$ holds. For any $j \in J(i)$ with $\alpha_{ij}(x_i) \neq 0$

we have $x_i \in x_{ij} + U_i$, hence $x_{ij} \in V$ which implies $y_{ij} \in W$. Then

$$x_i = f_i(x_i) = \sum_{j \in J(i)} \alpha_{ij}(x_i) y_{ij} \in W,$$

contradicting that $x_i \in V$. Therefore $x_0 \in T(x_0)$ holds.

Fan's fixed point theorem immediately implies the theorems of Kakutani [3] ($E = R^n$), Tychonoff [4] ($T(x) = \{f(x)\}$ for a continuous function $f: K \to K$), as well as the fundamental Brouwer fixed point theorem used in the proof above.

REFERENCES

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