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THE FIRST INITIAL-BOUNDARY VALUE PROBLEM FOR SOME NONLINEAR TIME DEGENERATE PARABOLIC EQUATIONS

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ABSTRACT. Consider the nonuniformly parabolic operator

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} - c(x,t,u)u_t + d(x,t)u,$$

where u, a^{ij}, b^i, c, d are bounded, real-valued functions defined on a domain $D=\Omega\times[0,T]\subset R^{n+1}$. Assume that $c(x,t,u)$ is Lipschitz continuous in $|\cdot|_\alpha^D$ of $C_\alpha(D)$, and that $c(x,t,u)\geqq 0$ on D. Sufficient conditions on c are found which guarantee existence of a unique solution $u\in\bar{C}_{2+\alpha}$ to the first initial-boundary value problem $Lu=f(x,t)$, $u=\psi$, on the normal boundary of D, where $\psi\in\bar{C}_{2+\alpha}$. Existence is proved by direct application of a fixed point theorem due to Schauder using existence of a solution to the linear problem as well as a priori estimates.

1. **Introduction.** We shall be concerned with existence of a solution to the first initial-boundary value problem for the second-order nonlinear nonuniformly parabolic operator

$$(1.1)\qquad Lu = \sum_{i,j=1}^{n} a^{ij}u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} - c(x,t,u)u_t + d(x,t)u,$$

where u and all coefficients of L are real-valued functions defined for $(x,t)=(x_1,\cdots,x_n,t)$ in an $(n+1)$-dimensional, bounded, convex domain D. Subscripts will be used to denote differentiation.

We will assume that L is parabolic; that is,

$$(1.2)\qquad \sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \geqq \gamma\,|\xi|^2 > 0$$

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for some $\gamma>0$ and for any real vector $\xi\neq 0$. Assume that the coefficient $c(x, t, u)\geqq 0$ but is not necessarily bounded away from zero. Since $c(x, t, u)$ may be zero for some $(x, t)\in D$, L may be a degenerate parabolic operator. Note that c may be a function of u as well as of x and t.

The author has already proved the existence of a unique solution to the first initial-boundary value problem for the linear equation

$$(1.3)\quad Mu = \sum_{i,j=1}^{n} a^{ij}u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} - c(x, t)u_t + d(x, t)u = f(x, t)$$

in [**4**]. Existence of a solution to the nonlinear problem is proved with the aid of a fixed point theorem due to Schauder. We shall also use the fact that we can obtain a solution to the linear initial-boundary value problem together with a priori estimates obtained in [**4**].

This partial differential equation arises in applications to studies dealing with fluid flow through porous media. One such application [**3**] deals with diffusion and filtration of lipid-protein complexes and oxygen through the arterial wall and into surrounding tissue. Historically, the porosity coefficient c has been assumed to be a positive constant, which is an average obtained in studying a healthy artery. However, tissue, as well as blood, is a heterogeneous composite of materials. In the case of a nonhealthy artery (as in atherosclerosis), it is precisely this heterogeneity which is of interest. The coefficient of porosity $c(x, t, u)$ is more accurately and usefully described as a coefficient of space, time, and pressure.

2. **Notation and the linear first initial-boundary value problem.** We shall use the same notation as A. Friedman [**2**, p. 40] to describe the domain D. As stated in the introduction, D is a bounded, convex, $(n+1)$-dimensional domain in R^{n+1}, where $(x, t)=(x_1, \cdots, x_n, t)$ represents a variable point in R^{n+1}. ∂D denotes the boundary of D. Thus, $\partial D=\bar{B}+B_T+S$, where B is a domain in $R^n\times\{0\}$, B_T $(T>0)$ is a domain in $R^n\times\{T\}$, and S is a manifold, not necessarily connected, in $R^n\times(0, T]$. $S+\bar{B}$ therefore denotes the parabolic, or normal, boundary of D.

Let $D_\tau=D\cap(R^n\times(0, \tau))$, $B_\tau=D\cap(R^n\times\{\tau\})$, and $S_\tau=S\cap(R^n\times(0, \tau])$. Assume that for each $\tau\in(0, T)$, B_τ is a domain. Then, for every (x, τ) in D, $0<\tau<T$, if $S(x, \tau)=D_\tau+B_\tau$, then $(S(x, \tau))^{-}-S(x, \tau)=\bar{B}+S_\tau$, where $(S(x, \tau))^{-}$ denotes the closure of $(S(x, \tau))$. Also, assume that the following is a condition on D: There exists a simple continuous curve α connecting B to B_T along which the t-coordinate is nondecreasing.

Introduce the metric d defined by

$$d(P, Q) = [|x - \bar{x}|^2 + |t - \bar{t}|]^{1/2}$$

where $P=(x, t)$, $Q=(\bar{x}, \bar{t})$, and $|x|=(\sum_{i=1}^{n} x_i^2)^{1/2}$. Hölder continuity of a function f is defined with respect to the metric d.

Suppose $\alpha \in (0, 1)$. Then let

$$|u|_0^D = \sup_D |u|, \qquad \bar{H}_\alpha^D(u) = \sup_{P,Q\in D} \frac{|u(P) - u(Q)|}{d(P, Q)^\alpha}, \qquad |\bar{u}|_\alpha^D = |u|_0^D + \bar{H}_\alpha^D(u).$$

Then, $\bar{C}_\alpha(D)=\{u\colon u\colon \bar{D}\to R, |\bar{u}|_\alpha^D<\infty\}$ is a Banach space with norm $|\bar{\cdot}|_\alpha^D$; see [**2**, pp. 62–63].

Denote by D_x^m any partial derivative of order m with respect to the variables $x_1, \cdots, x_n$ and let $D_t=\partial/\partial t$. If $D_x u$, $D_x^2 u$, $D_t u$ exist in D, then we define

$$|\bar{u}|_{2+\alpha}^D = |\bar{u}|_\alpha^D + \sum |(D_x u)^-|_\alpha^D + \sum |(D_x^2 u)^-|_\alpha^D + |(D_t u)^-|_\alpha^D,$$

where the sums are taken over all partial derivatives of the indicated order. Let

$$\bar{C}_{2+\alpha}(D) = \{u\colon u\colon \bar{D} \to R, |\bar{u}|_{2+\alpha}^D < \infty\}.$$

Then, $\bar{C}_{2+\alpha}(D)$ is a Banach space with norm $|\bar{\cdot}|_{2+\alpha}^D$; see [**2**]. When there is no confusion, we will drop the D from $|\bar{\cdot}|_\alpha^D$, $|\bar{\cdot}|_{2+\alpha}^D$.

Definition 2.1. We say that D has property $(\bar{E})$ if for every point Q of $\bar{S}$, there exists an $(n+1)$-dimensional neighborhood V such that $V\cap\bar{S}$ can be represented, for some i $(1\leqq i\leqq n)$, in the form

$$x_i = h(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t),$$

where h, $D_x h$, $D_x^2 h$, $D_t h$ are Hölder continuous of exponent α.

Definition 2.2. If D has property $(\bar{E})$ and if the functions $D_x D_t h$, $D_t^2 h$ of the local representations of $\bar{S}$ exist and are continuous functions, then we say D has property (E').

Definition 2.3. A function ψ defined on $\bar{B}+S$ is said to belong to $\bar{C}_{2+\alpha}(D)$ if there exist functions Ψ in $\bar{C}_{2+\alpha}(D)$ such that $\Psi=\psi$ on $\bar{B}+S$. Then $|\bar{\psi}|_{2+\alpha}^D$ is defined by

$$|\bar{\psi}|_{2+\alpha}^D = \inf_\Psi |\bar{\Psi}|_{2+\alpha}^D,$$

where the infimum is taken over all $\Psi \in \bar{C}_{2+\alpha}(D)$ which coincide with ψ on $\bar{B}+S$.

The following notation is one of convenience. If u is a bounded real-valued function on a subset S of R^{n+1}, define

$$M(u; S) = \sup\{u(x, t)\colon (x, t) \in S\}$$

and

$$m(u; S) = \inf\{u(x, t)\colon (x, t) \in S\}.$$

In [4], the author solves the linear first initial-boundary value problem

$$\begin{aligned} Nu &= \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t) u_{x_i} - c(x,t) u_t + d(x,t) u \\ &= f(x,t) \quad \text{on } D + B_T, \\ u &= \psi \quad \text{on } \bar{B} + S \end{aligned} \tag{2.1}$$

without assuming that c has a positive minimum in D. We now state this theorem precisely.

THEOREM 2.4. *Assume that a^{ij} is constant for each i, j, that all coefficients of N, defined in* (2.1) *are of class $C^{1,1}(\bar{D})$, $u \in C(\bar{D})$, $u \in C^3(D)$, and $a^{11}\lambda^2+b^1\lambda \geqq 1$ for some $\lambda>0$. Suppose, further, that the coefficients of N are uniformly Hölder continuous* (*exponent* α) *in D, $|(a^{ij})^{-}|_\alpha \leqq \bar{K}_1$, $|(b^i)^{-}|_\alpha \leqq \bar{K}_1$, $|\bar{c}|_\alpha \leqq \bar{K}_1$, $|\bar{d}|_\alpha \leqq \bar{K}_1$, that $m(c;\, \bar{B}+S) \geqq \mu > 0$ while $m(c;\, D) \geqq 0$, that $|\bar{f}|_\alpha < \infty$, and that* (1.2) *holds. If D has property (E'), $\psi \in \bar{C}_{2+\alpha}(D)$, and $N\psi = f$ on ∂B, then there exists a unique solution u of the first initial-boundary value problem* (2.1) *and, furthermore, $u \in \bar{C}_{2+\alpha}(D)$.*

3. **A priori estimates.** Our proof for the nonlinear case requires the use of an a priori estimate for a solution to the linear problem. The following theorem was proved in [4].

THEOREM 3.1. *Suppose that the conditions of Theorem* 2.4 *hold, together with the added restriction that $m(c;\, D) \geqq \mu > 0$. Then there exists a constant $\bar{K}$ depending only on $\bar{K}_1$, K_2, α, and D such that if u is a solution to $Nu=f$, with $u=\psi$ on $\bar{B}+S$, and if $u \in \bar{C}_{2+\alpha}(D)$, then*

$$|\bar{u}|_{2+\alpha} \leqq \mu^{-1/2} \bar{K}(|\bar{\psi}|_{2+\alpha} + |\bar{f}|_\alpha). \tag{3.1}$$

The technique which was used in obtaining the solution in Theorem 2.4 involved perturbing the coefficient c by $1/k$ and considering the problem

$$\begin{aligned} N^k u = Nu - (1/k) u_t &= f \quad \text{on } D + B_T, \\ u &= \psi \quad \text{on } \bar{B} + S. \end{aligned} \tag{3.2}$$

By Friedman's work, we were guaranteed a unique solution $u_k \in \bar{C}_{2+\alpha}(D)$. We were able to show that the sequence $\{u_k\}$ obtained in this manner is Cauchy in the Banach space $\bar{C}_{2+\alpha}(D)$ and does, in fact, converge to the unique solution of the first initial-boundary value problem (2.1).

These u_k, incidentally, satisfy

$$|(u_k)^{-}|_{2+\alpha} \leqq k^{1/2} \bar{K}(|\bar{\psi}|_{2+\alpha} + |\bar{f}|_\alpha). \tag{3.3}$$

If $|\bar{\psi}|_{2+\alpha}+|\bar{f}|_\alpha>0$, there is some u_K such that $|(u-u_K)^{-}|_{2+\alpha}<|\bar{\psi}|_{2+\alpha}+|\bar{f}|_\alpha$. Since u_K satisfies (3.1), we may conclude that

$$\begin{aligned}|\bar{u}|_{2+\alpha} &= |(u-u_K+u_K)^{-}|_{2+\alpha} \leqq |(u-u_K)^{-}|_{2+\alpha} + |(u_K)^{-}|_{2+\alpha} \\ &\leqq (|\bar{\psi}|_{2+\alpha}+|\bar{f}|_\alpha) + K^{1/2}\bar{K}(|\bar{\psi}|_{2+\alpha}+|\bar{f}|_\alpha) \\ &= (1+K^{1/2}\bar{K})(|\bar{\psi}|_{2+\alpha}+|\bar{f}|_\alpha).\end{aligned}$$

We have just proved the following theorem.

THEOREM 3.2. *Assume all the conditions of Theorem 2.4. Then the solution u of the first initial-boundary value problem* (2.1) *satisfies*

$$|\bar{u}|_{2+\alpha} \leqq \bar{M}(|\bar{\psi}|_{2+\alpha}+|\bar{f}|_\alpha), \tag{3.4}$$

for some constant $\bar{M}$ which is independent of $m(c; D)$, provided $|\bar{\psi}|_{2+\alpha}+|\bar{f}|_\alpha>$ 0.

4. **The nonlinear problem.** We now proceed to solve the first initial-boundary value problem (L defined as in (1.1)):

$$\begin{aligned} Lu &= f(x,t), \quad \text{on } D+B_T, \\ u &= \psi, \quad \text{on } \bar{B}+S. \end{aligned} \tag{4.1}$$

For v a fixed element of $\bar{C}_{2+\alpha}(D)$, consider the linear problem

$$\begin{aligned} L_v u &= \sum_{i,j=1}^{n} a^{ij}u_{x_ix_j} + \sum_{i=1}^{n} b^i u_{x_i} - c(x,t,v)u_t + du \\ &= f(x,t), \quad \text{on } D+B_T, \\ u &= \psi, \quad \text{on } \bar{B}+S. \end{aligned} \tag{4.2}$$

Assuming that as a function of x and t, $c(x,t,v)$ satisfies the hypotheses of Theorem 2.4, we obtain a solution $u\in\bar{C}_{2+\alpha}(D)$ to the initial-boundary value problem (4.2). u obviously depends upon v. Hence, we define $\phi:\bar{C}_{2+\alpha}(D)\to\bar{C}_{2+\alpha}(D)$ by $u=\phi(v)$ is the unique solution to the initial-boundary value problem (4.2).

Under appropriate conditions on c, we will show that

(i) $\phi: A\to A$, where A is some closed convex subset of $\bar{C}_{2+\alpha}(D)$;

(ii) ϕ is continuous in $|\bar{\cdot}|^D_{2+\alpha}$ on A.

We will then be able to apply the Schauder fixed point theorem of [**2**, p. 189] to obtain an element $u\in\bar{C}_{2+\alpha}(D)$ such that $\phi(u)=u$. Then u will be a solution to (4.1) and we will have proved the desired existence theorem.

An argument similar to that given by Friedman in [**2**] allows us to assume that $\psi\equiv 0$ on $\bar{B}+S$. We will assume that $c(x,t,u)$ is Lipschitz continuous with respect to u in the $|\bar{\cdot}|_\alpha$ norm. That is, if $(x,t)\in D$

and $u_1, u_2 \in C_\alpha(D)$, then

$$|(c(x, t, u_1) - c(x, t, u_2))^-|_\alpha^D \leqq \eta\, |(u_1 - u_2)^-|_\alpha^D$$

for some positive constant η.

Let $A=\{u: |\bar{u}|_{2+\alpha}^D \leqq 2\bar{M}|\bar{f}|_\alpha^D,\ u=0 \text{ on } \bar{B}+S\}$ where f is the forcing function in (4.2) and $\bar{M}$ is the constant obtained in Theorem 3.2. A is clearly a closed convex subset of $\bar{C}_{2+\alpha}(D)$.

Now suppose that $v \in A$ and $h \in [0, 1]$. Then $\phi(hv)$ satisfies $L_{hv}\phi(hv)=f$, $\phi(hv)=0$ on $\bar{B}+S$. We note that $L_{hv}\phi(hv)=f$ can be rewritten in the form

$$L_v\phi(v) = L_v\phi(v) - L_{hv}\phi(hv) + f \equiv F(v). \tag{4.3}$$

Hence, $\phi(v)$ satisfies:

$$\begin{aligned} L_v\phi(v) &= F(v) \quad \text{in } D + B_T, \\ \phi(v) &= 0 \qquad \text{on } \bar{B} + S. \end{aligned} \tag{4.4}$$

Using (4.3), we see that

$$\begin{aligned} |(F(v))^-|_\alpha^D &\leqq |(L_v\phi(v) - L_{hv}\phi(hv))^-|_\alpha^D + |\bar{f}|_\alpha^D \\ &\leqq K\, |(\phi(v) - \phi(hv))^-|_{2+\alpha}^D \\ &\quad + |(c(x, t, v) - c(x, t, hv))^-|_\alpha^D\, |([\phi(hv)]_t)^-|_\alpha^D + |\bar{f}|_\alpha^D. \end{aligned}$$

But, $|([\phi(hv)]_t)^-|_\alpha^D \leqq K_1$ for some $K_1>0$ and for all $h \in [0, 1]$. This follows from the proof of Theorem 6.4 of [**4**, pp. 53–54]. $\phi(v)-\phi(hv)$ satisfies

$$\begin{aligned} L_v(u) &= [c(x, t, v) - c(x, t, hv)][\phi(hv)]_t \quad \text{in } D + B_T, \\ u &= 0 \quad \text{on } \bar{B} + S. \end{aligned} \tag{4.5}$$

Applying Theorem 3.2 to the system (4.5), we have the estimate

$$|(\phi(v) - \phi(hv))^-|_{2+\alpha}^D \leqq \bar{M}\, |(c(x, t, v) - c(x, t, hv))^-|_\alpha^D\, |([\phi(hv)]_t)^-|_\alpha^D.$$

Therefore,

$$\begin{aligned} |(F(v))^-|_\alpha^D &\leqq K\, |(c(x, t, v) - c(x, t, hv))^-|_\alpha^D\, |([\phi(hv)]_t)^-|_\alpha^D \\ &\quad + |(c(x, t, v) - c(x, t, hv))^-|_\alpha^D\, |([\phi(hv)]_t)^-|_\alpha^D + |\bar{f}|_\alpha^D \\ &\leqq (K + 1)\eta\, |(v - hv)^-|_\alpha^D\, K_1 + |\bar{f}|_\alpha^D \\ &\leqq (K + 1)\eta K_1\, |1 - h|\, |\bar{v}|_\alpha^D + |\bar{f}|_\alpha^D. \end{aligned}$$

We have thus proved the inequality

$$|(F(v))^-|_\alpha^D \leqq (K + 1)\eta K_1\, |1 - h|\, |\bar{v}|_\alpha^D + |\bar{f}|_\alpha^D. \tag{4.6}$$

Applying Theorem 3.2 to the system (4.4), we have the estimate

$$|(\phi(v))^{-}|^{D}_{2+\alpha} \leqq \bar{M}\,|(F(v))^{-}|^{D}_{\alpha}. \tag{4.7}$$

Inequalities (4.6) and (4.7) yield

$$|(\phi(v))^{-}|^{D}_{2+\alpha} \leqq [(K+1)\eta K_1\,|1-h|\,|\bar{v}|^{D}_{\alpha} + |\bar{f}|^{D}_{\alpha}]\bar{M}.$$

But $v \in A$ implies $|\bar{v}|^{D}_{\alpha} \leqq |\bar{v}|^{D}_{2+\alpha} \leqq 2\bar{M}|\bar{f}|^{D}_{\alpha}$. Choose h so that $2(K+1)\eta K_1|1-h|\bar{M} \leqq 1$. Thus, $|(\phi(v))^{-}|^{D}_{2+\alpha} \leqq 2\bar{M}|\bar{f}|^{D}_{\alpha}$. We conclude that ϕ maps A to A.

It is easy to show that ϕ is continuous with respect to $|\bar{\cdot}|^{D}_{2+\alpha}$ in A. Note that if v_0 and v are in A, $\phi(v_0)-\phi(v)$ satisfies

$$\begin{aligned} L_{v_0}(u) &= [c(x,t,v_0) - c(x,t,v)][\phi(v)]_t \quad \text{in } D + B_T, \\ u &= 0 \quad \text{on } \bar{B} + S. \end{aligned} \tag{4.8}$$

Applying Theorem 3.2 to the system (3.8) we obtain the estimate

$$|(\phi(v) - \phi(v_0))^{-}|^{D}_{2+\alpha} \leqq \bar{M}\,|(c(x,t,v_0) - c(x,t,v))^{-}|^{D}_{\alpha}\,|([\phi(v)]_t)^{-}|^{D}_{\alpha}. \tag{4.9}$$

Recall that $|([\phi(v)]_t)^{-}|^{D}_{\alpha} \leqq K_1$. This uniform bound, together with Lipschitz continuity of $c(x,t,u)$ with respect to u, and (4.9) give

$$|(\phi(v) - \phi(v_0))^{-}|^{D}_{2+\alpha} \leqq \bar{M}K_1\eta\,|(v_0 - v)^{-}|^{D}_{\alpha}. \tag{4.10}$$

Thus, ϕ is continuous with respect to $|\bar{\cdot}|_{2+\alpha}$ in A.

A fixed point theorem due to Schauder [**2**, p. 189] can be used to conclude that there exists a function $u \in A$ for which $\phi(u)=u$. We have thus proved the following theorem.

THEOREM 4.1 *Assume that the conditions of Theorem* 2.4 *hold for the operator L, defined in* (1.1), *and suppose also that $c(x,t,u)$ is Lipschitz continuous with respect to u in the $|\bar{\cdot}|_\alpha$ norm. If D has property* (E'), $\psi \in \bar{C}_{2+\alpha}(D)$, *and $L\psi=f$ on ∂B, then there exists a solution u, of the nonlinear first initial-boundary value problem* (4.1) *and furthermore,* $u \in \bar{C}_{2+\alpha}(D)$.

THEOREM 4.2. *The solution u obtained in Theorem* 4.1 *is unique.*

PROOF. The uniqueness of the solution to the nonlinear problem (4.1) follows from uniqueness of the solution to the linear problem of Theorem 2.4 by a well-known argument using the mean-value theorem. See for example a paper by A. P. Calderón [**1**, p. 35], or extend in a straightforward manner the argument given by A. Friedman [**2**, p. 41].

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