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REGULAR AND BAER RINGS

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ABSTRACT. The (von Neumann) regular Baer rings representable as the full ring $E(G)$ of all endomorphisms of an abelian group G are characterized. It is also shown that a countable regular Baer ring is Artinian semisimple.

All the rings that we consider are associative with identity and all the modules are unitary left modules. A ring R is called (von Neumann) regular if each $a \in R$ satisfies $a=axa$ for some $x \in R$. R is strongly regular if each a satisfies $a=xa^2=a^2x$ for some $x \in R$. A ring R is said to be Baer if every left (equivalently, right) annihilator ideal is generated by an idempotent [3]. The additive group of a ring R is denoted by R^+. Let G be an abelian group. Then $E(G)$ will denote the ring of all endomorphisms of G, G_t the subgroup of all elements of finite order and, for each prime p, G_p the subgroup of all elements of order a power of p. A subgroup S is pure in G if $nS=S\cap nG$ for every positive integer n. A subring S is pure in R if S^+ is pure in R^+. If $\{A_i\}$, $i \in I$, are rings or groups, then $\prod_{i\in I} A_i$ ($\bigoplus_{i\in I} A_i$) denotes their direct product (sum). Q denotes the field of rational numbers and $Z(p)$ the field of integers modulo a prime p.

1. **Strongly regular Baer rings.** First note that a regular ring is strongly regular if and only if it has no nonzero nilpotent elements or, equivalently, all its idempotents are central. Thus for a strongly regular ring R the lattice $L(R)$ of all principal left ideals of R is isomorphic to the Boolean algebra $B(R)$ of all idempotents in R. Moreover the lattice of all left (=right) ideals of R is isomorphic to the lattice of all ideals of $B(R)$. Hence from a well-known theorem of M. H. Stone we get: A strongly regular ring R is Baer$\Leftrightarrow B(R)$ is complete$\Leftrightarrow$the maximal ideal space of R is extremely disconnected. Since a complete Boolean algebra cannot be countably infinite, the last remark implies that a countable strongly regular Baer ring is Artinian semisimple. Actually a more general result holds, as indicated below.

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THEOREM 1. *A regular Baer ring of cardinality* $<c$, *the continuum is Artinian semisimple.*

PROOF. It is enough if we show that R satisfies the A.C.C. for principal left ideals. Suppose $Re_1 \subset Re_2 \subset \cdots$ is a strictly increasing chain where $e_n = e_n^2$ for all n. Clearly $e_n e_{n+1} = e_n$ for all n. Define inductively $f_1 = e_1$, $f_{n+1} = f_n + e_{n+1} - e_{n+1} f_n$. It is then easy to check that $Re_n = Rf_n$ and $f_n f_{n+1} = f_{n+1} f_n = f_n$ for all n. Hence, by induction, $f_n f_m = f_m f_n = f_n$ for all $m > n$. Define $g_n = f_n - f_{n+1}$. Then $g_n^2 = g_n$ for all n and $g_m g_n = 0$ if $m \neq n$. Note that $g_n \neq 0$, since $Rf_n = Re_n \neq Re_{n+1} = Rf_{n+1}$. To each nonempty subset S of natural numbers, choose an idempotent h_S in R such that the right annihilator of $\{g_i : i \in S\}$ is $(1-h_S)R$. Clearly $g_i h_S = g_i$ for all $i \in S$, and $(1-h_S)g_k = g_k$ (i.e., $h_S g_k = 0$) for all $k \notin S$. Note that $h_S g_k = 0$ implies $g_k h_S g_k = 0$ so that $g_k h_S \neq g_k$. Hence if S and T are distinct subsets of natural numbers, then $h_S \neq h_T$. This contradicts the hypothesis that R has cardinality less then c. Hence the result.

2. **Regular and Baer endomorphism rings.** In this section we completely characterize those regular Baer rings which appear as the full rings of endomorphisms of abelian groups. These rings turn out to be 'almost' self-injective. Since our work here is a continuation of [**6**], we freely make use of the results and terminology of [**6**].

LEMMA 1. *Let S be a pure subgroup of $\prod Z(p)^+$, where p runs over a set of primes, such that $\bigoplus Z(p)^+ \subseteq S \subseteq \prod Z(p)^+$. Then the following are true*:

(i) *if $\rho: S \to S$ is a Z-morphism then, for any element $x = \langle \cdots, x_p, \cdots \rangle$ in S, $x\rho = \langle \cdots, x_p\rho_p, \cdots \rangle$ where $\rho_p = \rho | Z(p)$*;

(ii) *every endomorphism of S uniquely extends to an endomorphism of $\prod Z(p)^+$.*

LEMMA 2. *Let P be a set of primes and R a pure subring of $\prod_{p \in P} Z(p)$ containing $\bigoplus Z(p)$ and the identity. Then $R \cong \mathrm{Hom}_Z(R^+, R^+)$.*

PROOF. Since $R \cong \mathrm{Hom}_R(R, R)$, it is enough to show that every Z-endomorphism of R is an R-morphism. But this is immediate from Lemma 1(i) and the fact that, for all $p \in P$, $\rho_p = \rho | Z(p)$ is an R-morphism.

As a corollary we get a description of strongly regular endomorphism rings (see Problems 44 and 50 in [**1**]).

COROLLARY. *Let R be strongly regular. Then $R = E(G)$ for some abelian group G if and only if either $R \cong \prod F_i$, where the F_i are non-isomorphic prime fields or R is a regular subring with identity of $\prod_{p \in P} Z(p)$ containing $\bigoplus Z(p)$, where P is a set of primes.*

PROOF. $R=E(G)$ regular implies (see [5]) that *either* $G=D\oplus E$, where D is a direct sum of copies of Q^+ and $E=\bigoplus E_p$, each E_p being a direct sum of copies of $Z(p)^+$ *or* G is a pure subgroup of $\prod G_p$ and each G_p is a direct sum of copies of $Z(p)^+$. If G contains a summand of the form $A\oplus A$, then it is easy to construct an endomorphism $\rho\neq 0$ such that $\rho^2=0$. Since R has no nonzero nilpotent elements, we get, in the first case, that $D\cong Q^+$ and $E_p\cong Z(p)^+$ and, in the second case, that $G_p=Z(p)^+$. This proves the necessity. The sufficiency follows from Lemma 2 if one notes that a regular subring of $\prod Z(p)$ is pure.

We are now ready to prove the main theorem of this paper.

THEOREM 2. *Let R be a regular Baer ring. Then $R\cong E(G)$ for some abelian group G if and only if R is a ring direct sum, $R=R_1\oplus R_2$, where*

(i) *R_2 is left self-injective being isomorphic to $\prod_{i\in N} L_i$ where, for each $i\in N$, $L_i\cong \mathrm{Hom}_{F_i}(V_i, V_i)$ with V_i a vector space of dimension $\geqq 2$ over a prime field F_i and $F_i\not\cong F_j$ if $i\neq j$, and*

(ii) *R_1 is a (commutative) regular subring containing all the idempotents of the ring $\prod_{p\in T} Z(p)$, where T is a set of primes and no p in T is the characteristic of any of the fields F_i. Moreover, $R_1=0$ if any of the fields F_i has characteristic zero.*

PROOF. Suppose $R\cong E(G)$ for some abelian group G. Then, by Remark 4.2 of [6], *either* (a) $G=D\oplus S$, D torsion-free divisible and $S=\bigoplus S_p$ is elementary, p running over a set of primes, *or* (b) G is reduced, G_t is elementary and G/G_t is divisible. In the case (a), D and S_p can be considered as vector spaces over the fields Q and $Z(p)$ respectively and their Z-endomorphisms are vector space morphisms. Consequently,

$$R\cong \mathrm{Hom}_Q(D, D)\oplus \prod_p \mathrm{Hom}_{Z(p)}(S_p, S_p)$$

which is in the required form. Consider the case (b). We can write $\bigoplus G_p\subseteq G\subseteq \prod G_p$. Since every endomorphism of G uniquely extends to an endomorphism of $\prod G_p$, we consider $R=E(G)$ as a subring of $E(\prod G_p)\cong \prod E(G_p)$ containing $\bigoplus E(G_p)$. Let T be the set of all primes p for which G_p is cyclic. Define $S_1=\bigoplus_{p\in T} G_p$ and $S_2=\bigoplus_{p\notin T} G_p$. If A_i $(i=1, 2)$ is the closure of S_i under the n-adic topology of G, then, by Theorem 3.3 of [6], $G=A_1\oplus A_2$. Clearly the A_i are invariant under all the endomorphisms of G so that $R\cong E(A_1)\oplus E(A_2)$.

Let $R_1=E(A_1)$. Since $(A_1)_p=Z(p)^+$, R_1 is isomorphic to a subring of $\prod_{p\in T} Z(p)$ so that R_1 is commutative. Since R_1 is Baer, by Theorem 3.3 and Remark 3.5 of [6], R_1 contains all the idempotents of $\prod_{p\in T} Z(p)$.

Let $R_2=E(A_2)$. Now R_2 is regular and Baer and hence is left continuous, by [6]. Let I be a nonzero two sided ideal of R_2. We wish to show that I

contains a nonzero nilpotent element. Since $(R_2)_p \simeq E(G_p)$ and $\bigoplus (R_2)_p \subseteq R_2 \subseteq \prod (R_2)_p$, and since each $(R_2)_p$ is an elementary abelian p-group, I contains a nonzero ρ satisfying $p\rho = 0$ for some prime p. Then Im $\rho \subset$ the elementary abelian p-group G_p and, composing ρ with a projection of G_p onto a cyclic summand of G_p, get a $\lambda \in I$ with Im λ a cyclic summand, say $G = \operatorname{Im} \lambda \oplus C$. If $\ker \lambda \cap G_p = 0$, then G_p would isomorphically map into Im λ and hence G_p is cyclic, a contradiction. Thus $\ker \lambda \cap G_p \neq 0$. Let μ be an endomorphism of G given by $\mu | C = 0$ and $(\mu | \operatorname{Im} \lambda) =$ a monomorphism from Im λ to $(G_p \cap \ker \lambda)$. Then $0 \neq \lambda\mu \in I$ and $(\lambda\mu)^2 = 0$. Thus every nonzero two sided ideal of the left continuous regular ring R_2 contains a nonzero nilpotent element. By Theorem 3 of [7], R_2 is therefore left self-injective. Then Theorem 4.6 of [6] implies that $R_2 \simeq \prod_{i \in N} L_i$ with $L_i \simeq \operatorname{Hom}_{F_i}(V_i, V_i)$, where the F_i's are prime fields with nonzero characteristic and $F_i \not\simeq F_j$ if $i \neq j$.

Thus in both cases, $R = R_1 \oplus R_2$, where the R_i's have the required form. Moreover, $R_1 \neq 0$ only in the case (b) wherein none of the F_i has characteristic zero.

Conversely, let $R = R_1 \oplus R_2$ be in the required form. Since the F_i are prime fields, $\operatorname{Hom}_{F_i}(V_i, V_i) = \operatorname{Hom}_Z(V_i, V_i)$ and since the F_i are nonisomorphic, $R_2 \simeq E(H)$ where $H = \bigoplus V_i^+$. Now by Lemma 2, $R_1 = E(R_1^+)$. If $R_1 \neq 0$, then each F_i has, by hypothesis, characteristic $p_i > 0$ so that, in view of our assumption on the set T,

$$\operatorname{Hom}_Z(R_1^+, H) = 0 = \operatorname{Hom}_Z(H, R_1^+).$$

Then $R = E(G)$, where $G = H \oplus R_1^+$.

REMARK. If R is regular Baer and $R \simeq E(G)$, then R need not be self-injective. This is equivalent to saying that the ring R_1 of Theorem 2 need not be self-injective. To see this, let P be the set of all prime integers >0 and let A be the subring of $\prod_{p \in P} Z(p)$ containing $\bigoplus Z(p)$ and the identity such that $A/(\bigoplus Z(p)) \simeq Q$, the field of rational numbers (see [2]). Clearly A is regular. Let B be the subring generated by A and all idempotents of $\prod Z(p)$. Since $B = \sum Ae$, where e runs over all the idempotents of $\prod Z(p)$, it is easy to see that B is regular (and Baer). Now $B \neq \prod Z(p)$, since B does not contain the element $x = \langle \cdots, x_p, \cdots \rangle$, where $x_p^2 \equiv -1 \pmod p$, if p is of the form $4n+1$ and $x_p = 0$, otherwise. [Note that, to each $y \in A$, there exists a rational number s/t with $(s, t) = 1$ and $ty_p \equiv s \pmod p$ for all except a finite number of primes $p \in P$.] Since B is distinct from its injective envelope $\prod Z(p)$, B is not self-injective.

By remark 4.3 of [6], we have the following

COROLLARY. *Let G be an abelian group. Then $E(G)$ is regular Baer if and only if either $G = D \oplus E$, where $D \neq 0$ is torsion-free divisible and*

E is elementary or G is reduced and $G=A\oplus B$, *where*

(i) *A is a fully invariant subgroup of* $\prod_{p\in P} E_p$ *with* E_p *being an elementary p-group and P a set of primes, and*

(ii) *B is a pure subgroup of* $\prod_{p\in P'} Z(p)^+$ *with the property that every endomorphic image and also each subgroup closed under the n-adic topology of B is a direct summand of B, where P′ is another set of primes with* $P\cap P'=\varnothing$.

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