PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 2, April 1974

REMARKS ON "OPERATORS WITH INVERSES SIMILAR TO THEIR ADJOINTS"

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ABSTRACT. A result in the cited paper [7] which purports to furnish a sufficient condition for an operator T on a Hilbert space $\mathfrak{H}$ for which $T^{-1}S=ST^*$ with $0 \notin \mathrm{cl}(W(S))$ to be unitary is shown to be false. Various conditions under which such a result does hold are explored.

Let $\boldsymbol{B}(\mathfrak{H})$ denote the Banach algebra of all bounded linear operators on the complex Hilbert space $\mathfrak{H}$. Denote by $\sigma(T)$ the spectrum of $T \in \boldsymbol{B}(\mathfrak{H})$ and by $W(T)=\{(Tx, x) \mid \|x\|=1\}$ the numerical range of T. $r(T)=\sup\{|\lambda| \mid \lambda \in \sigma(T)\}$ is the spectral radius of T and $\omega(T)=\sup\{|\lambda| \mid \lambda \in W(T)\}$ is the numerical radius of T. We have $r(T) \leqq \omega(T) \leqq \|T\| \leqq 2\omega(T)$. T is normaloid if $\omega(T)=\|T\|$ from which follows in fact that $r(T)=\|T\|$ [**5**, Problem 173]. T is convexoid if $\mathrm{cl}(W(T))=\mathrm{co}\,\sigma(T)$. A normaloid operator need not be convexoid nor need a convexoid operator be normaloid [**5**, Problem 174]. Normal operators, and more generally hyponormal operators, are both normaloid and convexoid [**1**], [**8**].

1. The following assertion appears as Corollary 3 in [**7**]:

(*) *Let $S, T \in \boldsymbol{B}(\mathfrak{H})$ with T invertible. If* (i) $T^{-1}S=ST^*$ *with* $0 \notin \mathrm{cl}(W(S))$, *and* (ii) *$T$ is normaloid, then T is unitary.*

To construct a suitable counterexample to (*) first note that by Corollary 2 (and its converse) of [**7**], (i) holds iff T is similar to a unitary. Consequently (i) implies that $r(T)=1$ so that T is then normaloid iff $\|T\|=1$. Thus assume $\mathfrak{H}$ is separable and let U_0 be the bilateral shift defined by $U_0e_k=e_{k+1}$, $k \in \boldsymbol{Z}$, where $\{e_k\}_{k=-\infty}^{\infty}$ is an orthonormal basis for $\mathfrak{H}$. U_0 is unitary and $\sigma(U_0)=\{\lambda \mid |\lambda|=1\}$. Let Q be the positive operator defined by $Qe_k=\alpha_ke_k$, $k \in \boldsymbol{Z}$, with $\alpha_j=\alpha_{-j}^{-1}=2$ for $j>0$ and $\alpha_0=1$. Set $T_0=Q^{-1}U_0Q$, then $T_0e_k=e_{k+1}$ for $k\neq 0, -1$ and $T_0e_k=\frac{1}{2}e_{k+1}$ otherwise. Clearly $\|T_0\|=1$, but T_0 is not unitary.

Received by the editors August 21, 1973.

AMS (MOS) subject classifications (1970). Primary 47A10, 47B20; Secondary 47B47.

Key words and phrases. Hilbert spaces, similarities of unitaries, normaloid operators, convexoid operators, numerical range, multiplicative commutators.

[1] This research was supported in part by NSF Grant GP-23392.

Since in the above example $\sigma(T_0)=\sigma(U_0)$ and $\|T_0\|=1$, we see that $\mathrm{cl}(W(T_0))=\{\lambda|\ |\lambda|\leqq 1\}$ so that T_0 is also convexoid. Hence (*) is also false when (ii) is replaced by the assumption that T is convexoid. On the other hand if (ii) is replaced by the stronger hypothesis that T is normal, then the revised assertion which we label (**) is obviously true, since a normal whose spectrum lies on the unit circle is unitary. Actually (**) is also an immediate corollary of a correct form of (*), namely

THEOREM 1. *Let $T, S \in \boldsymbol{B}(\mathfrak{H})$ with T invertible. If*
(i) $T^{-1}S=ST^*$ *with* $0 \notin \mathrm{cl}(W(S))$,
(ii)′ *T is either convexoid or normaloid,*
(iii)′ *T^{-1} is either convexoid or normaloid, then T is unitary.*

PROOF. As in [7], (i) implies T is similar to a unitary so that $r(T)=1$. If additionally T is convexoid, then $W(T)\subset\Delta=\{\lambda|\ |\lambda|\leqq 1\}$; likewise if T is normaloid $\|T\|=r(T)=1$ by a remark in the introduction and again $W(T)\subset\Delta$. Because T^{-1} is also similar to a unitary, (iii)′ implies $W(T^{-1})\subset\Delta$ as well. A theorem of Stampfli [9, Theorem 1] may now be applied to yield the conclusion that T is unitary.

REMARK. If one of the conditions (ii)′ and (iii)′ is a normaloid condition, the application of the Stampfli theorem may be replaced by that of a simpler theorem of Donoghue [4] that asserts $W(T)\subset\Delta$ and $\|T^{-1}\|\leqq 1$ implies T is unitary.

COROLLARY 1. *Let $T, S \in \boldsymbol{B}(\mathfrak{H})$ with T invertible. If* (i) *holds and if T is hyponormal, then T is unitary.*

PROOF. T is hyponormal and invertible iff T^{-1} is hyponormal. Moreover every hyponormal is normaloid [1].

2. In the event that $\mathfrak{H}$ is finite dimensional (*) is indeed true when T is either normaloid or convexoid. We state this as

THEOREM 2 [2]. *Let $\mathfrak{H}$ be finite dimensional. If the hypotheses of* (*) *hold with* (ii) *replaced by* (ii)′ (*of Theorem* 1), *then T is unitary.*

PROOF. (See [2].) As above T is similar to a unitary and consequently is diagonizable. Moreover $W(T)\subset\Delta$ and $\sigma(T)\subset\partial W(T)$. Thus the eigenspaces of T are reducing and mutually orthogonal [6]. From these facts, it follows that T is unitarily equivalent to a diagonal unitary and so is in fact unitary.

3. We may employ (**) to produce a simple direct proof of Theorem 2 of [7]. Recall that a unitary V is cramped if $\sigma(V)$ is contained in an open arc of the unit circle of length π.

THEOREM 3 ([7], [2]). *Suppose $T \in \boldsymbol{B}(\mathfrak{H})$ is invertible. If V is a cramped unitary such that $V = TVT^*$, then T is unitary.*

PROOF. By taking inverses and adjoints in the relation $V = TVT^*$, we obtain $V = T^*VT$. Hence $V = T^*TVT^*T$. Since V is a cramped unitary, it is readily seen that $0 \notin \mathrm{cl}(W(V))$. Thus the conditions of $(**)$ are satisfied with T replaced by the positive (hence normal) operator T^*T. Therefore T^*T is unitary. Hence $T^*T = I$, i.e. T is unitary.

4. A slightly stronger version of $(**)$ is possible, namely

THEOREM 4. *If $T, S \in \boldsymbol{B}(\mathfrak{H})$ are invertible and if* (i)″ $T^{-1}S = ST^*$ *with* $0 \notin W(S)$, *and* (ii)″ *T is normal, then T is unitary.*

Note. It is $W(S)$ and not $\mathrm{cl}(W(S))$ which occurs in (i)″.

PROOF. From (i)″ we find $(T^*T)^{-1} = T^{*-1}ST^*S^{-1}$. Because T is normal so are T^{*-1} and $(T^*T)^{-1}$ and moreover these two operators commute. Consequently the conditions of the generalized Marcus-Thompson multiplicative commutator theorem which appears as Theorem 2 in [3] are satisfied. Thus either $0 \in W(S)$ or T^{*-1}, and therefore T^*, commutes with S. In view of (i)″ it follows that $T^{-1} = T^*$.

REMARK. Theorem 4 clearly permits an improvement of Theorem 3 replacing the condition that the unitary V be cramped by the weaker condition $0 \notin W(V)$. We do not know whether Theorem 1 may be similarly improved.

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