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STARLIKE FUNCTIONS

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ABSTRACT. Let $\mathscr{S}^*[\alpha]$ denote the class of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ analytic in $|z|<1$ and for which $|zf'(z)/f(z)-1|<1-\alpha$ for $|z|<1$ and $\alpha \in [0, 1)$. Sharp results concerning coefficients, distortion, and the radius of convexity are obtained. Furthermore, it is shown that $\sum_{n=2}^{\infty} [(n-\alpha)/(1-\alpha)]|a_n|<1$ is a sufficient condition for $f(z) \in \mathscr{S}^*[\alpha]$.

1. **Introduction.** Suppose that $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is analytic and $\operatorname{Re}\{zf'(z)/f(z)\}>\alpha$ for $|z|<1$ and $\alpha \in [0, 1)$, then $f(z)$ is called starlike of order α, denoted by $f(z) \in \mathscr{S}_\alpha^*$. It was shown by Schild [7] that for $f(z) \in \mathscr{S}_\alpha^*$ the domain of values of $\{zf'(z)/f(z)\}$ is the circle with line segment from $(1+(2\alpha-1)|z|)/(1+|z|)$ to $(1-(2\alpha-1)|z|)/(1-|z|)$ as a diameter. In this paper we consider a subclass of $\mathscr{S}_\alpha^*$ consisting of those $f(z)$ for which $|zf'(z)/f(z)-1|<1-\alpha$ for $|z|<1$ and denote it by $\mathscr{S}^*[\alpha]$. Sharp results concerning coefficients, distortion, and the radius of convexity are obtained. Furthermore, it is shown that $\sum_{n=2}^{\infty} [(n-\alpha)/(1-\alpha)]|a_n|\leqq 1$ is a sufficient condition for $f(z)$ to be in $\mathscr{S}^*[\alpha]$. Results about $\sum_{n=2}^{\infty} n|a_n|\leqq 1$ have previously been the subject of papers by Goodman [3], MacGregor [4], and Schild [6].

2. **Coefficient theorems.**

THEOREM 1. *Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ and suppose that*

$$\sum_{n=2}^{\infty} [(n-\alpha)/(1-\alpha)]\,|a_n| \leqq 1;$$

then $f(z) \in \mathscr{S}^*[\alpha]$ for $\alpha \in [0, 1)$.

PROOF. Let $|z|=1$, then

$$\begin{aligned} |zf'(z)-f(z)| - (1-\alpha)\,|f(z)| &= \left|\sum_{n=2}^{\infty} (n-1)a_n z^n\right| - (1-\alpha)\left|z+\sum_{n=2}^{\infty} a_n z^n\right| \\ &\leqq \sum_{n=2}^{\infty} (n-\alpha)\,|a_n| - (1-\alpha) \leqq 0 \end{aligned}$$

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by the hypothesis. Hence $|zf'(z)/f(z)-1|<1-\alpha$ for $|z|<1$ by the maximum modulus theorem. We note that $f(z)=z-[(1-\alpha)/(n-\alpha)]z^n$ is an extremal function with respect to the theorem since $|zf'(z)/f(z)-1|=1-\alpha$ for $z=1$, $\alpha\in[0, 1)$, and $n=2, 3, \cdots$. We also observe that the converse to the theorem is false in that $f(z)=ze^{(1-\alpha)z}\in\mathcal{S}^*[\alpha]$ but

$$\sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n|=\sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}\frac{(1-\alpha)^{n-1}}{(n-1)!}>2e^{1-\alpha}-1>1 \quad \text{for all } \alpha\in[0, 1).$$

For $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}_\alpha^*$ it has been shown [7] that the sharp inequality $|a_n|\leqq\prod_{k=2}^{n}(k-2\alpha)/(n-1)!$ holds for $n=2, 3, \cdots$ and is attained by $f(z)=z(1-z)^{-2(1-\alpha)}$. It is interesting to note that $(1-\alpha)/(n-1)$ appears as a factor of the upper bound for each $|a_n|$. For our class it is the upper bound.

THEOREM 2. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}^*[\alpha]$, then $|a_n|\leqq(1-\alpha)/(n-1)$ for $n=2, 3, \cdots$ and $\alpha\in[0, 1)$.*

PROOF. If $f(z)\in\mathcal{S}^*[\alpha]$, then $|zf'(z)/f(z)-1|<1-\alpha$ and since the absolute value vanishes for $z=0$ we obtain

$$zf'(z)/f(z)=1+\omega(z) \tag{1}$$

where $\omega(z)=\sum_{k=1}^{\infty}\omega_kz^k$ is analytic and $|\omega(z)|<1-\alpha$ for $|z|<1$. From (1) we see that $zf'(z)-f(z)=f(z)\omega(z)$ or equivalently

$$\sum_{k=2}^{\infty}(k-1)a_kz^k=\left(z+\sum_{k=2}^{\infty}a_kz^k\right)\left(\sum_{k=1}^{\infty}\omega_kz^k\right). \tag{2}$$

Equating coefficients on both sides of (2) shows that

$$(n-1)a_n=\omega_{n-1}+\sum_{k=2}^{n-1}a_1\omega_{n-k} \quad \text{for } n=2, 3, \cdots$$

which implies that the coefficient a_n on the left side of (2) is dependent only on $a_2, a_3, \cdots, a_{n-1}$ on the right side of (2). Hence for $n\geqq2$

$$\sum_{k=2}^{n}(k-1)a_kz^k+\sum_{k=n+1}^{\infty}A_kz^k=\left(z+\sum_{k=2}^{n-1}a_kz^k\right)\omega(z) \tag{3}$$

for a proper choice of A_k. Squaring the moduli of both sides of (3) and integrating around $|z|=r<1$ we get, using the fact that $|\omega(z)|<1-\alpha$ for $|z|<1$,

$$\sum_{k=2}^{n}(k-1)^2|a_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|A_k|^2r^{2k}<\left(1+\sum_{k=2}^{n-1}|a_k|^2\right)(1-\alpha)^2.$$

Let $r \to 1$ and we find that

$$\sum_{k=2}^{n} (k-1)^2 |a_k|^2 \leqq \left(1 + \sum_{k=2}^{n-1} |a_k|^2\right)(1-\alpha)^2$$

or

$$(n-1)^2 |a_n|^2 \leqq (1-\alpha)^2 - \sum_{k=2}^{n-1} [(k-1)^2 - (1-\alpha)^2] |a_k|^2 \leqq (1-\alpha)^2$$

and it follows that $|a_n| \leqq (1-\alpha)/(n-1)$. This proof is based on a technique found in Clunie and Keogh [2]. For sharpness, consider the function

$$f(z) = z \exp[(1-\alpha)/(n-1)]z^{n-1} = z + [(1-\alpha)/(n-1)]z^n + \cdots.$$

3. **Distortion theorems.** In order to obtain distortion properties of $f(z)$ and $f'(z)$ we need the representation given by the following lemma.

LEMMA 1. *$f(z) \in \mathscr{S}^*[\alpha]$ if and only if*

$$f(z) = z \cdot \exp\left\{\int_0^z \phi(t)\, dt\right\}$$

where $\phi(z)$ is analytic for $|z|<1$ and $|\phi(z)| \leqq 1-\alpha$ for $|z|<1$ and $\alpha \in [0, 1)$.

PROOF. The "only if" is easily obtained by integrating (1) with $\omega(z) = z\phi(z)$ while $f(z) \in \mathscr{S}^*[\alpha]$ follows from differentiation and simple manipulation.

THEOREM 3. *If $f(z) \in \mathscr{S}^*[\alpha]$, then*

(i) $$|z|\, e^{(\alpha-1)|z|} \leqq |f(z)| \leqq |z|\, e^{(1-\alpha)|z|},$$

and

(ii) $$(1 + (\alpha-1)|z|)e^{(\alpha-1)|z|} \leqq |f'(z)| \leqq (1 + (1-\alpha)|z|)e^{(1-\alpha)|z|}.$$

PROOF. Since

$$\left|\int_0^z \phi(t)\, dt\right| \leqq \int_0^{|z|} |\phi(t)|\, |dt| \leqq \int_0^{|z|} (1-\alpha)\, |dt| = (1-\alpha)\, |z|,$$

then (i) follows by virtue of Lemma 1, and (ii) by applying the triangle inequality to $f'(z) = (1 + z\phi(z)) \cdot f(z)/z$. Both parts of the theorem are sharp for $f(z) = ze^{(1-\alpha)z}$.

4. **The radius of convexity.** It is well known that every univalent function maps $|z| < 2-\sqrt{3} = 0.267\cdots$ onto a convex region and that for the class of starlike functions $\mathscr{S}_0^*$ this estimate cannot be improved since the extremal function for the class of univalent functions, the Koebe function $K(z) = z/(1-e^{i\theta}z)^2$, is also starlike. Here we determine the exact

radius of convexity of $\mathscr{S}^*[\alpha]$ as a function of α for each $\alpha \in [0, 1)$. In particular, it is shown that for $\mathscr{S}^*[0]$ the radius of convexity is $(3-\sqrt{5})/2=0.382\cdots$.

THEOREM 4. *Suppose that $f(z) \in \mathscr{S}^*[\alpha]$; then $f(z)$ maps $|z|<r$ onto a convex domain for*

$$\text{(i)} \qquad r = (3-\sqrt{5})/(2-2\alpha), \quad \text{if } \alpha \in [0, \alpha_0],$$

and

$$\text{(ii)} \qquad r = [((-2\alpha^2+\alpha-1)+2\alpha(6-6\alpha+\alpha^2)^{1/2})/(1+3\alpha-4\alpha^2)]^{1/2}, \quad \text{if } \alpha \in [\alpha_0, 1),$$

where $\alpha_0=1-(1+\sqrt{6})(3\sqrt{5}-5)/10=0.411\cdots$. The result is sharp.

PROOF. As a notational convenience let $\beta=1-\alpha$. In **[1]** it is shown that if $\phi(z)$ is analytic for $|z|<1$ with $|\phi(z)|\leqq 1$, then

$$|\phi'(z)| \leqq (1-|\phi(z)|^2)/(1-|z|^2).$$

If, however, $|\phi(z)|\leqq\beta$ then we may apply the previous result to $\phi(z)/\beta$ and obtain

$$|\phi'(z)| \leqq (\beta-|\phi(z)|^2\beta^{-1})/(1-|z|^2). \tag{4}$$

Let $\omega(z)=-z\phi(z)$ in (1); then after taking the logarithmic derivative of both sides we have

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}+1\right\} = \operatorname{Re}\left\{(1-z\phi(z)) - \frac{z^2\phi'(z)+z\phi(z)}{1-z\phi(z)}\right\}. \tag{5}$$

Regrouping and then applying (4) and the triangle inequality to the right side of (5) gives us

$$\begin{aligned}\operatorname{Re}\{zf''(z)/f'(z)+1\} &\geqq 1-(|z\phi(z)|\,(1+(1-|z\phi(z)|)^{-1}) \\ &\qquad + |z|^2\,|\phi'(z)|\,(1-|z\phi(z)|)^{-1}) \\ &\geqq 1-(|z\phi(z)|\,(2-|z\phi(z)|)(1-|z|^2) \\ &\qquad + |z|^2(\beta-|\phi(z)|^2\beta^{-1})) \\ &\qquad \cdot((1-|z\phi(z)|)(1-|z|^2))^{-1}.\end{aligned} \tag{6}$$

For $f(z)$ to be convex in $|z|<r$ it suffices to have $\operatorname{Re}\{zf''(z)/f'(z)+1\}>0$ in $|z|<r$, see **[5]**. In our case, (5) will certainly be positive when the right side of (6) is positive which is whenever

$$\begin{aligned}\beta\,|z\phi(z)|\,(2-|z\phi(z)|)(1-|z|^2)+|z|^2(\beta^2-|\phi(z)|^2) \\ < \beta(1-|z\phi(z)|)(1-|z|^2).\end{aligned} \tag{7}$$

In order to discuss (7) let us consider the function

$$p(x) = [(1 - r^2)\beta + 1]x^2 - 3(1 - r^2)\beta x - (\beta^2 r^2 - \beta(1 - r^2))$$

where $r=|z| \in [0, 1)$ and $x=|z\phi(z)| \in [0, r\beta]$. Clearly

$$p'(x) = 2[(1 - r^2)\beta + 1]x - 3(1 - r^2)\beta = 0$$

for $x=x_0=3(1-r^2)\beta/(2((1-r^2)\beta+1))$ and $p''(x)>0$. Hence, for each β, $p(x)$ is a parabola opening upward. Thus an investigation of (7) now reduces to trying to determine some relationship among β, r, and x so that $p(x)>0$. In order to do so we explore the geometric significance of the two cases: (a) $p'(r\beta)\leqq 0$ and (b) $p'(r\beta)\geqq 0$.

(a) If $p'(r\beta)\leqq 0$, then $r\beta\leqq x_0$ and so $p(x)\geqq p(r\beta)$ for $x \in [0, r\beta]$. Now, let $p(r\beta)$ be considered as a function of r with β held constant, then (7) will be satisfied when

$$p(r\beta) = \beta(1 - r^2)(1 - 3\beta r + \beta^2 r^2) > 0$$

which is exactly when $r<(3-\sqrt{5})/(2\beta)$. This result, however, is valid only for those β for which $r\beta\leqq x_0$ or equivalently, for β satisfying

$$2\sqrt{5}\beta^2 + (2\sqrt{5} - 6)\beta + (15 - 7\sqrt{5}) \geqq 0.$$

Hence for $\beta \in [\beta_0, 1]$ where $\beta_0=(1+\sqrt{6})(3\sqrt{5}-5)/10$. The result is proved sharp by choosing the function $f(z)=ze^{\beta z}$ for $\beta \in [\beta_0, 1]$.

(b) On the other hand, $p'(r\beta)\geqq 0$ implies $r\beta\geqq x_0$ and $p(x)\geqq p(x_0)$ for $x \in [0, r\beta]$. Again let β be fixed. Now (7) will be satisfied when

$$p(x_0) = -\beta((5\beta - 4\beta^2)r^4 + 2(2\beta^2 - 3\beta + 2)r^2 + (5\beta - 4)) \cdot (4((1 - r^2)\beta + 1))^{-2} > 0$$

which is whenever

$$r < r_0 = [(-(2\beta^2 - 3\beta + 2) + 2(1 - \beta)(\beta^2 + 4\beta + 1)^{1/2})/(5\beta - 4\beta^2)]^{1/2}$$

provided that $r\beta\geqq x_0$, that is, for $\beta \in (0, \beta_0]$.

We conclude with an existence proof for a sharp function. Suppose that $\psi(z)=\beta(z-z_0)/(1-z_0z)$ where z_0 is real; then

$$z_0 = (\psi(z) - \beta z)/(z\psi(z) - \beta)$$

and so

$$\psi'(z) = \beta(1 - z_0^2)/(1 - z_0 z)^2 = (\beta^2 - \psi^2(z))/(\beta(1 - z^2)).$$

If we now let $zf'(z)/f(z)=1-z\psi(z)$, then

$$\begin{aligned} zf''(z)/f'(z) + 1 \\ &= (((1 - z^2)\beta + 1)(z\psi(z))^2 - 3(1 - z^2)\beta(z\psi(z)) \\ &\quad - (\beta^2 z^2 - \beta(1 - z^2)))/(\beta(1 - z\psi(z))(1 - z^2)) = 0 \end{aligned}$$

when $z=r_0$ and $z\psi(z)=r_0\psi(r_0)=x_0$. Since $x_0\leqq r_0\beta$ we have

$$r_0\psi(r_0) = r_0\beta(r_0 - z_0)/(1 - r_0z_0) \leqq r_0\beta$$

and so $(r_0-z_0)/(1-r_0z_0)\leqq 1$ which implies that $|z_0|\leqq 1$. Hence $|\psi(z)|\leqq\beta$ for $|z|<1$ and so by Lemma 1

$$f(z) = z \cdot \exp\left\{\int_0^z (-\beta(t - z_0)/(1 - z_0t))\, dt\right\} \in \mathscr{S}^*[1 - \beta].$$

References

1. C. Carathéodory, *Theory of functions of a complex variable*. Vol. 2, Chelsea, New York, 1954. MR **16**, 346.

2. J. Clunie and F. R. Keogh, *On starlike and convex schlicht functions*, J. London Math. Soc. **35** (1960), 229–233. MR **22** #1682.

3. A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. **8** (1957), 598–601. MR **19**, 260.

4. T. H. MacGregor, *A class of univalent functions*, Proc. Amer. Math. Soc. **15** (1964), 311–317. MR **28** #2206.

5. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952. MR **13**, 640.

6. A. Schild, *On a class of functions schlicht in the unit circle*, Proc. Amer. Math. Soc. **5** (1954), 115–120. MR **15**, 694.

7. ———, *On starlike functions of order* α, Amer. J. Math. **87** (1965), 65–70. MR **30** #4929.

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