PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 2, April 1974

ON THE WEIGHT OF A TOPOLOGICAL SPACE

R. E. HODEL

ABSTRACT. The main result in this paper states that the weight of a regular space is the product of three cardinal functions, namely the Lindelöf degree, the pluming degree, and the point separating weight.

1. **Introduction.** In recent years there has been considerable interest in cardinal functions, no doubt due in large part to Arhangel'skiĭ's recent solution [3] of Alexandroff's problem. (See Comfort's paper [6] for a survey of cardinal invariants; the fundamental tract on cardinal functions is Juhász [9].) The main result of this paper states that the weight of a regular space is the product of three cardinal functions. This result is motivated by the following three theorems, due respectively to Ponomarev [13], Arhangel'skiĭ [4], and Nagata [12]. (1) Every Lindelöf p-space with a point-countable base has a countable base. (2) If X is a p-space and $\mathcal{N}$ is a net for X, then there is a base $\mathcal{B}$ for X with $|\mathcal{B}|\leqq|\mathcal{N}|$. (3) Every paracompact p-space with a point-countable separating open cover is metrizable. We also study conditions under which the weight of an infinite Hausdorff space X is not greater than $|X|$.

Throughout this paper $\mathfrak{m}$ and $\mathfrak{n}$ denote cardinal numbers, σ denotes an ordinal number, and $|A|$ denotes the cardinality of a set A. All regular spaces are T_1 and all p-spaces are T_1 and completely regular.

2. **The main theorem.** We begin with some definitions and known results which are needed to state and prove the main theorem. Let X be a set, let $\mathcal{G}$ be a cover of X. The cover $\mathcal{G}$ is said to be *separating* if given distinct points p and q in X, there is some G in $\mathcal{G}$ such that $p\in G$, $q\notin G$. For p in X, the *order of p with respect* to $\mathcal{G}$, denoted $\operatorname{ord}(p,\mathcal{G})$, is the cardinality of the set $\{G \text{ in } \mathcal{G} : p\in G\}$.

Now let X be a topological space. A collection $\mathcal{N}$ of subsets of X is a *net* for X if for each point p in X and each neighborhood R of p, there is some N in $\mathcal{N}$ such that $p\in N\subseteq R$. The *weight of* X, denoted $w(X)$, is $\aleph_0\cdot\mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal number which arises as the cardinality of some base for X. The *Lindelöf degree* of X, denoted $L(X)$,

Received by the editors April 28, 1973.
AMS (MOS) subject classifications (1970). Primary 54A25.
Key words and phrases. Weight, Lindelöf degree, p-space, separating open cover.

is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that every open cover of X has a subcover of cardinality $\leqq \mathfrak{m}$.

Now suppose X is a T_1 space. Let $\mathfrak{m}$ be the smallest cardinal such that X has a separating open cover $\mathscr{S}$ with $\operatorname{ord}(p, \mathscr{S}) \leqq \mathfrak{m}$ for all p in X. (Since X is T_1, it is easy to see that $\mathfrak{m}$ exists and $\mathfrak{m} \leqq |X|$.) The *point separating weight* of X, denoted $\operatorname{psw}(X)$, is $\aleph_0 \cdot \mathfrak{m}$.

We now extend the concept of a p-space to higher cardinality by introducing the pluming degree of a regular space. Our definition is based on an internal characterization of p-spaces due to Burke [**5**]. A collection $\{\mathscr{G}_\alpha : \alpha \in A\}$ of open covers of a topological space X is a *pluming* for X if the following holds: if $p \in G_\alpha \in \mathscr{G}_\alpha$ for all α in A, then (a) $C_p = \bigcap_{\alpha \in A} \bar{G}_\alpha$ is compact; (b) $\{\bigcap_{\alpha \in F} \bar{G}_\alpha : F$ a finite subset of $A\}$ is a "base" for C_p in the sense that given any open set R containing C_p, there is a finite subset F of A with $\bigcap_{\alpha \in F} \bar{G}_\alpha \subseteq R$. Before defining the pluming degree we need the following existence result.

PROPOSITION 2.1. *Let X be a regular space. Then X has a pluming $\{\mathscr{G}_\alpha : \alpha \in A\}$ with $|A| \leqq w(X)$.*

PROOF. Let $\mathscr{B} = \{B_\alpha : \alpha \in A\}$ be a base for X with $|A| \leqq w(X)$. Let $D = \{(\alpha, \beta)$ in $A \times A : \bar{B}_\alpha \subseteq B_\beta\}$, and for each (α, β) in D let $\mathscr{G}(\alpha, \beta) = \{X - \bar{B}_\alpha, B_\beta\}$. Then $\{\mathscr{G}(\alpha, \beta) : (\alpha, \beta) \in D\}$ is a pluming for X with $|D| \leqq w(X)$. $\square$

For a regular space X, the *pluming degree* of X, denoted $p(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that X has a pluming $\{\mathscr{G}_\alpha : \alpha \in A\}$ with $|A| = \mathfrak{m}$. By the above proposition, $p(X) \leqq w(X)$. Also, it follows from Burke's theorem [**5**] that for a completely regular space X, $p(X) = \aleph_0$ if and only if X is a p-space.

The following two lemmas are used in the proof of the main theorem. (See [**11**], [**7**], and [**8**].)

LEMMA 2.2 (MIŠČENKO). *Let X be a set, let $\mathfrak{m}$ be an infinite cardinal, let $\mathscr{S}$ be a collection of subsets of X such that* $\operatorname{ord}(p, \mathscr{S}) \leqq \mathfrak{m}$ *for all p in X, and let H be a subset of X. Then the cardinality of the set of all finite minimal covers of H by elements of $\mathscr{S}$ does not exceed* $\mathfrak{m}$.

LEMMA 2.3 (HOLSZTYŃSKI). *Let X be a Hausdorff space, let $\mathscr{N}$ be a net for X with $|\mathscr{N}| \leqq \mathfrak{m}$. Then there is an open cover $\mathscr{U}$ of X with $|\mathscr{U}| \leqq \mathfrak{m}$ such that given distinct points p and q in X, there exists U in $\mathscr{U}$ such that $p \in U$, $q \notin \bar{U}$.*

THEOREM 2.4. *Let X be a regular space. Then*

$$w(X) = L(X) \cdot p(X) \cdot \operatorname{psw}(X).$$

PROOF. Clearly $L(X)\cdot p(X)\cdot \operatorname{psw}(X)\leqq w(X)$. Suppose, then, that $L(X)\cdot p(X)\cdot \operatorname{psw}(X)=\mathfrak{m}$, and let us construct a base $\mathscr{B}$ for X with $|\mathscr{B}|\leqq\mathfrak{m}$. Let $\{\mathscr{G}_\alpha:\alpha\in A\}$ be a pluming for X with $|A|\leqq\mathfrak{m}$. Since $L(X)\leqq\mathfrak{m}$, we may assume that $|\mathscr{G}_\alpha|\leqq\mathfrak{m}$ for each α in A. Let $\mathscr{G}=\bigcup_{\alpha\in A}\mathscr{G}_\alpha$ and let $\mathscr{H}$ be all finite intersections of elements of $\mathscr{G}$. Note that $|\mathscr{H}|\leqq\mathfrak{m}$. Let $\mathscr{S}$ be a separating open cover of X such that $\operatorname{ord}(p,\mathscr{S})\leqq\mathfrak{m}$ for all p in X. We may assume that $X\in\mathscr{S}$, and hence for any subset H of X there is at least one finite minimal cover of H by elements of $\mathscr{S}$, namely $\{X\}$.

First we shall prove that $|\mathscr{S}|\leqq\mathfrak{m}$. For each H in $\mathscr{H}$ let $\{\mathscr{S}(H,\sigma): 0\leqq\sigma<\mathfrak{n}_H\leqq\mathfrak{m}\}$ be all finite minimal covers of H by elements of $\mathscr{S}$ (use Miščenko's lemma), and let $\mathscr{S}'=\bigcup\{\mathscr{S}(H,\sigma):H\in\mathscr{H}, 0\leqq\sigma<\mathfrak{n}_H\}$. We shall show that $\mathscr{S}\subseteq\mathscr{S}'$, from which it follows that $|\mathscr{S}|\leqq\mathfrak{m}$. Let $S_0\in\mathscr{S}$, and let $p\in S_0$. For each α in A choose G_α in $\mathscr{G}_\alpha$ such that $p\in G_\alpha$, and let $C_p=\bigcap_{\alpha\in A}\bar{G}_\alpha$. Recall that C_p is compact. Construct a subcollection $\mathscr{S}_0$ of $\mathscr{S}$ which covers C_p as follows: $S_0\in\mathscr{S}_0$, and for each q in (C_p-S_0) choose an element of $\mathscr{S}$ which contains q but not p. Let $\mathscr{S}_1$ be a finite subcollection of $\mathscr{S}_0$ which covers C_p. Let F be a finite subset of A such that $\bigcap_{\alpha\in F}\bar{G}_\alpha\subseteq\bigcup\mathscr{S}_1$, and let $H=\bigcap_{\alpha\in F}G_\alpha$. Let $\mathscr{S}_2$ be a minimal subcollection of $\mathscr{S}_1$ which covers H, and note that $S_0\in\mathscr{S}_2$. Now $\mathscr{S}_2=\mathscr{S}(H,\sigma)$ for some $\sigma<\mathfrak{n}_H$, and so $S_0\in\mathscr{S}'$.

Next let us construct a net $\mathscr{N}$ for X with $|\mathscr{N}|\leqq\mathfrak{m}$. Let $\mathscr{N}=\{H-W: H\in\mathscr{H}, W=\varnothing$ or W a finite union of elements of $\mathscr{S}\}$. Clearly $|\mathscr{N}|\leqq\mathfrak{m}$. To see that $\mathscr{N}$ is a net, let $p\in X$, let R be an open neighborhood of p. For each α in A choose G_α in $\mathscr{G}_\alpha$ such that $p\in G_\alpha$, and let $C_p=\bigcap_{\alpha\in A}\bar{G}_\alpha$. First suppose $C_p\subseteq R$. Choose a finite subset F of A such that $\bigcap_{\alpha\in F}\bar{G}_\alpha\subseteq R$. Then $H=\bigcap_{\alpha\in F}G_\alpha$ is an element of $\mathscr{N}$ and $p\in H\subseteq R$. Next, assume that $C_p-R\neq\varnothing$. Let $\mathscr{S}_0$ be a finite subcollection of $\mathscr{S}$ which covers C_p-R such that $p\notin\bigcup\mathscr{S}_0=W$. Now $C_p\subseteq R\cup W$, so there is a finite subset F of A such that $\bigcap_{\alpha\in F}\bar{G}_\alpha\subseteq W\cup R$. Let $H=\bigcap_{\alpha\in F}G_\alpha$. Then $H-W$ is an element of $\mathscr{N}$ such that $p\in H-W\subseteq R$.

Finally, we construct a base $\mathscr{B}$ for X with $|\mathscr{B}|\leqq\mathfrak{m}$. Since X has a net $\mathscr{N}$ with $|\mathscr{N}|\leqq\mathfrak{m}$, it follows from Lemma 2.3 that there is an open cover $\mathscr{U}$ of X with $|\mathscr{U}|\leqq\mathfrak{m}$ such that given distinct points p, q in X, there exists U in $\mathscr{U}$ such that $p\in U$, $q\notin\bar{U}$. We may assume that $\mathscr{U}$ is closed under finite intersections and $X\in\mathscr{U}$. Let $\mathscr{B}=\{H\cap U:H\in\mathscr{H}, U\in\mathscr{U}\}$. It is clear that $\mathscr{B}$ is an open collection with $|\mathscr{B}|\leqq\mathfrak{m}$, and so it remains to show $\mathscr{B}$ a base. Let $p\in X$, let R be an open neighborhood of p. For each α in A choose G_α in $\mathscr{G}_\alpha$ such that $p\in G_\alpha$, and let $C_p=\bigcap_{\alpha\in A}\bar{G}_\alpha$. If $C_p\subseteq R$, the result is clear. Suppose, then, that $C_p-R\neq\varnothing$. Let U and W be disjoint open sets such that $p\in U\in\mathscr{U}$ and $C_p-R\subseteq W$. Now $C_p\subseteq W\cup R$, so there is a finite subset F of A with $\bigcap_{\alpha\in F} \bar{G}_\alpha \subseteq W\cup R$. Hence

$$B = \left(\bigcap_{\alpha\in F} G_\alpha\right) \cap U$$

is an element of $\mathscr{B}$ such that $p \in B \subseteq R$. This completes the proof.

COROLLARY 2.5. *Every Lindelöf p-space with a point-countable separating open cover has a countable base.*

COROLLARY 2.6. *Let* $\mathfrak{m}$ *be infinite, let* X *be a regular space. Suppose* $X=\bigcup_{\alpha\in A} X_\alpha$, *where* $|A|\leqq\mathfrak{m}$ *and* $w(X_\alpha)\leqq\mathfrak{m}$ *for all* α *in* A. *Then* $w(X)\leqq\mathfrak{m}$ *if and only if* $p(X)\leqq\mathfrak{m}$.

PROOF. Suppose $p(X)\leqq\mathfrak{m}$. Clearly X has a net $\mathscr{N}$ with $|\mathscr{N}|\leqq\mathfrak{m}$. It easily follows that $L(X)\leqq\mathfrak{m}$, and by Lemma 2.3 X has a separating open cover $\mathscr{U}$ with $|\mathscr{U}|\leqq\mathfrak{m}$. Hence $w(X)\leqq\mathfrak{m}$. □

REMARK 2.7. Corollary 2.5 gives a generalization of the result by Ponomarev mentioned in §1. Corollary 2.6 is a slight extension of the beautiful Weight Addition Theorem of Arhangel'skiĭ **[4]**.

3. Conditions under which $w(X)\leqq|X|$.

It is well known that $w(X)\leqq 2^{|X|}$ for any infinite topological space X. However, as we now show, $w(X)\leqq|X|$ under fairly weak assumptions on X. The following result is basic.

THEOREM 3.1. *Let* X *be an infinite Hausdorff space. Then* $w(X)\leqq|X|$ *if and only if there is a cover* $\mathscr{K}$ *of* X *such that each* K *in* $\mathscr{K}$ *is compact and has a base consisting of not more than* $|X|$ *open sets.*

PROOF. If $w(X)\leqq|X|$, the existence of $\mathscr{K}$ is obvious. Assume, then, that $\mathscr{K}$ is a cover of X having the stated properties. Clearly we may assume $|\mathscr{K}|\leqq|X|$. For each K in $\mathscr{K}$ let $\{V(K,\sigma):0\leqq\sigma<\mathfrak{m}_K\leqq|X|\}$ be a base for K. Now X has a net $\mathscr{N}$ with $|\mathscr{N}|\leqq|X|$, so by Lemma 2.3 there is an open cover $\mathscr{U}$ of X with $|\mathscr{U}|\leqq|X|$ such that given distinct points p, q in X, there exists U in $\mathscr{U}$ such that $p\in U$, $q\notin\bar{U}$. We may assume that $\mathscr{U}$ is closed under finite intersections and $X\in\mathscr{U}$. Let $\mathscr{B}=\{U\cap V(K,\sigma): U\in\mathscr{U}, K\in\mathscr{K}, 0\leqq\sigma<\mathfrak{m}_K\}$. Then $\mathscr{B}$ is a base for X with $|\mathscr{B}|\leqq|X|$. □

REMARK 3.2. Let X be a topological space. Recall that X is of *point-countable type* **[2]** if each point of X belongs to a compact set having a countable base, and X is a *q-space* **[10]** if each point p of X has a countable collection $\{U(n,p):n=1,2,\cdots\}$ of open neighborhoods such that if $x_n\in U(n,p)$ for $n=1,2,\cdots$, then $\langle x_n\rangle$ has a cluster point. It is clear from Theorem 3.1 that for every infinite Hausdorff space X of point-countable type, $w(X)\leqq|X|$. (This result has recently been announced by Nagami.) It is not difficult to show that every regular q-space in which every countably compact space is compact is a space of point-countable type. We thus have the following corollary of Theorem 3.1. (See [**14**] for the definition of θ-refinable; a topological space X is *meta-Lindelöf* if every open cover has a point-countable open refinement. See [**1**].)

COROLLARY 3.3. *Let X be an infinite regular q-space which is either θ-refinable or meta-Lindelöf. Then $w(X) \leqq |X|$.*

Problem 3.4. Let $\mathfrak{m}$ be a cardinal, $\mathfrak{m} > \aleph_0$. Is there a countably compact regular space X such that $|X| = \mathfrak{m}$, $w(X) > \mathfrak{m}$? In a private communication to the author, W. W. Comfort constructed, for each cardinal $\mathfrak{m}$ such that $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, a countably compact completely regular space X with $|X| = \mathfrak{m}$ and $w(X) > \mathfrak{m}$.

REFERENCES

1. G. Aquaro, *Point-countable open coverings in countably compact spaces*, General Topology and its Relations to Modern Analysis and Algebra. II, Academia, Prague, 1966, pp. 39–41.

2. A. V. Arhangel'skiĭ, *Bicompact sets and the topology of spaces*, Trudy Moskov. Mat. Obšč. **13** (1965), 3–55=Trans. Moscow Math. Soc. 1965, 1–62. MR **33** #3251.

3. ———, *On the cardinality of bicompacta satisfying the first axiom of countability*, Dokl. Akad. Nauk SSSR **187** (1969), 967–970=Soviet Math. Dokl. **10** (1969), 951–955. MR **40** #4922.

4. ———, *On a class of spaces containing all metric and all locally bicompact spaces*, Mat. Sb. **67** (109) (1965), 55–88; English transl., Amer. Math. Soc. Transl. (2) **92** (1970), 1–39. MR **32** #8299; **42** #3.

5. D. K. Burke, *On p-spaces and wΔ-spaces*, Pacific J. Math. **35** (1970), 285–296. MR **43** #3986.

6. W. W. Comfort, *A survey of cardinal invariants*, General Topology and Appl. **1** (1971), no. 2, 163–199. MR **44** #7510.

7. V. V. Filippov, *On feathered paracompacta*, Dokl. Akad. Nauk SSSR **178** (1968), 555–558=Soviet Math. Dokl. **9** (1968), 161–164. MR **37** #3519.

8. W. Holsztyński, *Hausdorff spaces of minimal weight*, Dokl. Akad. Nauk SSSR **168** (1966), 508–509=Soviet Math. Dokl. **7** (1966), 667–668. MR **33** #3242.

9. I. Juhász, *Cardinal functions in topology*, Mathematical Centre, Amsterdam, 1971.

10. E. Michael, *A note on closed maps and compact sets*, Israel J. Math. **2** (1964), 173–176. MR **31** #1659.

11. A. S. Miščenko, *Spaces with point-countable bases*, Dokl. Akad. Nauk SSSR **144** (1962), 985–988=Soviet Math. Dokl. **3** (1962), 855–858. MR **25** #1537.

12. J. Nagata, *A note on Filippov's theorem*, Proc. Japan Acad. **45** (1969), 30–33. MR **39** #7557.

13. V. I. Ponomarev, *Metrizability of a finally compact p-space with a point-countable base*, Dokl. Akad. Nauk SSSR **174** (1967), 1274–1277=Soviet Math. Dokl. **8** (1967), 765–768. MR **35** #7298.

14. H. H. Wicke and J. M. Worrell, Jr., *Characterizations of developable spaces*, Canad. J. Math. **17** (1965), 820–830. MR **32** #427.

DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY, DURHAM, NORTH CAROLINA 27706