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SHORTER NOTES

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0-DIVISORS IN GROUP RINGS

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ABSTRACT. If G is any group with two finite subgroups H, K, $K \leqq G$, $(|H|, |K|)=1$, then RG has $\cap_\alpha \mathfrak{G}_\alpha \neq 0$, where $\mathfrak{G}$ is the augmentation ideal.

Let R be a commutative unitary ring of characteristic 0, and $\mathfrak{G}$ denote the augmentation ideal $\mathfrak{A}(G)$ of the group ring RG for a group G. If G is finite and not of prime-power order then, as J. Roseblade and R. Phillips have recently proved (unpublished), RG contains a 0-divisor congruent to 1 modulo $\mathfrak{G}$. Their proof depends heavily on properties of Schmidt groups. We give here a simple proof generalizing this result to the infinite case. For $T \leqq G$, let $\mathfrak{A}(T)$ be the left ideal in RT, generated by $\{t-1 \mid t \in T\}$.

THEOREM. *Let G be a group containing two finite subgroups H and K where $H \subseteq N_G(K)$ and $(|H|, |K|)=1$. Then $\mathfrak{A}(K) \cdot \mathfrak{A}(H) \cdot x=0$ for some $x \equiv 1 \bmod \mathfrak{G}$ in RG.*

PROOF. Let $y=\sum_{h \in H} h$, $z=\sum_{k \in K} k$, $|H|=m$, $|K|=n$, where $(m, n)=1$. Then $k \in K \Rightarrow (k-1)z=0$, and similarly $h \in H \Rightarrow (h-1)y=0$. Since $(m, n)=1$, there exist r, s in Z (and hence in R) such that $rm+sn=1$. Put $x=ry+sz$. If $\rho: RG \to R$ is the augmentation map, then $\rho(x)=r\rho(y)+s\rho(z)=rm+sn=1$, so that $x \equiv 1 \bmod \mathfrak{G}$, since $\mathfrak{G}=\text{kernel } \rho$. Also

$$\begin{aligned}\mathfrak{A}(K) \cdot \mathfrak{A}(H) \cdot x &= r \cdot \mathfrak{A}(K) \cdot \mathfrak{A}(H)y + s \cdot \mathfrak{A}(K) \cdot \mathfrak{A}(H)z \\ &= 0 + s \cdot \mathfrak{A}(K) \cdot \mathfrak{A}(H) \cdot z, \quad \text{since } \mathfrak{A}(H)y = 0, \\ &= s \cdot \mathfrak{A}(K) \cdot z \cdot \mathfrak{A}(H), \quad \text{since } H \subseteq N_G(K), \\ &= 0, \quad \text{since } \mathfrak{A}(K) \cdot z = 0.\end{aligned}$$

This proves the result. Q.E.D.

Since the existence of such 0-divisors easily implies that the intersection of all powers of the augmentation ideal is not 0, we have:

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COROLLARY. *If G has a finite subgroup which is not of prime-power order then* $\bigcap_\alpha \mathfrak{G}^\alpha \neq 0$.

PROOF. We may suppose that G is finite and not of prime-power order. It suffices to show that G has subgroups H, $K \neq 1$ of relatively prime order with $H \subseteq N_G(K)$.

Let P be a p-Sylow subgroup of G. If G has a normal p-complement K, take $P=H$. If not, there exists a subgroup $K \neq 1$ in P, such that $N_G(P)/C_G(P)$ is not a p-group. Take H to be a q-subgroup of $N_G(K)$ for some $q \neq p$.

With these subgroups H and K, we can now apply the Theorem and the comment above to complete the proof of the Corollary. Q.E.D.

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