PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 1, May 1974

FREE-LATTICE-LIKE SUBLATTICES OF FREE PRODUCTS OF LATTICES

G. GRÄTZER AND H. LAKSER[1]

ABSTRACT. Sublattices of a free lattice are known to satisfy three particular conditions. It is shown that certain sublattices of a free product of lattices satisfy the same three conditions.

1. **Introduction.** A free lattice is known to satisfy the following conditions:

(F1) $x\wedge y\leqq u\vee v$ implies that $x\leqq u\vee v$, or $y\leqq u\vee v$, or $x\wedge y\leqq u$, or $x\wedge y\leqq v$;

(F2) $u=x\vee y=x\vee z$ implies $u=x\vee(y\wedge z)$;

(F3) $u=x\wedge y=x\wedge z$ implies $u=x\wedge(y\vee z)$.

(F1) is in P. M. Whitman [**6**], and (F2), (F3) are in B. Jónsson [**3**].

These three conditions have proved of prime importance in working with free lattices. Indeed, a great many results about free lattices are derived solely from (F1), (F2), and (F3). For example, F. Galvin and B. Jónsson [**2**] used (F1) to investigate distributive sublattices of free lattices, and (F2), (F3) are used in B. Jónsson [**3**]; indeed they imply immediately that a free lattice cannot contain a nondistributive modular sublattice.

In this paper we show that certain sublattices of a free product of lattices satisfy (F1), (F2), and (F3).

2. **The free product.** The results presented in this section are those of [**1**] with several slight changes; we have no need of lattice polynomials, and it is more convenient to have upper and lower covers defined always as suggested by B. Jónsson [**4**]. We consequently proceed as follows. Let the lattice L be a free product of the family of (distinct) sublattices $(L_i \mid i\in I)$, and let $L^b=L\cup\{0_b, 1_b\}$ denote the bounded lattice with $0_b, 1_b\notin L$ and $0_b<x<1_b$ for all $x\in L$. For each $i\in I$ let $L_i^b=L_i\cup\{0_b, 1_b\}$. For each $a\in L$, $i\in I$, we have upper and lower i-covers $a^{(i)}, a_{(i)}\in L_i^b$. Indeed, for $i, j\in I$ define the homomorphisms $\varphi_{ji}, \varphi_j^i: L_j\to L_i^b$ by $x\varphi_{ji}=x\varphi_j^i=x$ if $j=i$, and $x\varphi_{ji}=0_b$, $x\varphi_j^i=1_b$ if $j\neq i$. Then the φ_{ji}, φ_j^i, $j\in I$, extend to respective

Presented to the Society, April 17, 1972; received by the editors October 16, 1972.

AMS (MOS) subject classifications (1970). Primary 06A20.

Key words and phrases. Free lattice, free product.

[1] The research of both authors was supported by the National Research Council of Canada.

homomorphisms φ_i, $\varphi^i: L \to L_i^b$, and we denote $a\varphi_i$ by $a_{(i)}$ and $a\varphi^i$ by $a^{(i)}$ for each $a \in L$. We say that $a_{(i)}$ is *proper* if $a_{(i)} \in L_i$, that is, if $a_{(i)} \neq 0_b$, and similarly $a^{(i)}$ is proper if $a^{(i)} \in L_i$, that is, if $a^{(i)} \neq 1_b$. Observe that if, under the convention of [**1**], $a_{(i)}$ (respectively $a^{(i)}$) exists then, under our present convention, $a_{(i)}$ (respectively $a^{(i)}$) is proper, and conversely.

We have the following lemma, which is a reformulation of Definition 3 and Theorem 1 of [**1**].

LEMMA 1. *Let L be a free product of the family of sublattices $(L_i \mid i \in I)$, and let $a, b, c, d \in L$. Then $a \wedge b \leqq c \vee d$ if and only if one of the following three conditions holds*:

(i) $a^{(i)} \wedge b^{(i)} \leqq c_{(i)} \vee d_{(i)}$ *for some* $i \in I$;

(ii) $a \wedge b \leqq c$ *or* $a \wedge b \leqq d$;

(iii) $a \leqq c \vee d$ *or* $b \leqq c \vee d$.

Since, for each $i \in I$, φ_i and φ^i are lattice homomorphisms extending the identity map $L_i \to L_i^b$, we have the following result and its dual.

LEMMA 2. *For each $i \in I$, $a \in L$ implies $a_{(i)} \leqq a$, and $x \in L_i$, $a \in L$, $x \leqq a$ imply $x \leqq a_{(i)}$.*

We also recall a result of [**5**].

LEMMA 3. *Let $u \in L$. Then u can be written as $u = u_0 \vee \cdots \vee u_{n-1}$, $n \geqq 1$, where the u_j, $j < n$, satisfy the following conditions*:

(i) $u_j \notin \bigcup (L_i \mid i \in I)$ *implies that* $u_j = a_j \wedge b_j$ *for some* $a_j, b_j \in L$ *with* $u_j < a_j$ *and* $u_j < b_j$;

(ii) *for each* $j < n$, $u_j \nleqq u_0 \vee \cdots \vee u_{j-1} \vee u_{j+1} \vee \cdots \vee u_{n-1}$;

(iii) *if* $j < n$ *and* $u_j \notin \bigcup (L_i \mid i \in I)$ *then, for all* $i \in I$, $u_j^{(i)} \nleqq u_{(i)}$;

(iv) *if, for* $j < n$, $u_j = a \wedge b$ *with* $a, b > u_j$ *then* $a \nleqq u$ *and* $b \nleqq u$;

(v) *if* $j, k < n$, $i \in I$, *and* $u_j, u_k \in L_i$, *then* $j = k$.

Indeed, any minimal representation (see [**5**]) of u yields such $u_0, \cdots, u_{n-1}$.

3. The theorem.

We now state and prove our result.

THEOREM. *Let L be a free product of the family of sublattices $(L_i \mid i \in I)$. For each $i \in I$ let K_i be a sublattice of L_i^b and let K be a sublattice of L such that $a_{(i)}, a^{(i)} \in K_i$ for all $a \in K$, $i \in I$. Let $n \in \{1, 2, 3\}$. If, for all $i \in I$, K_i satisfies* (Fn) *then K satisfies* (Fn).

PROOF. For each $i \in I$ let K_i satisfy (F1). Let $a, b, c, d \in K$ and let $a \wedge b \leqq c \vee d$. If, for some $i \in I$, $a^{(i)} \wedge b^{(i)} \leqq c_{(i)} \vee d_{(i)}$ then, since K_i satisfies (F1), we conclude that $a^{(i)} \leqq c_{(i)} \vee d_{(i)}$, or $b^{(i)} \leqq c_{(i)} \vee d_{(i)}$, or $a^{(i)} \wedge b^{(i)} \leqq c_{(i)}$, or $a^{(i)} \wedge b^{(i)} \leqq d_{(i)}$. By Lemma 2 and its dual it follows that $a \leqq c \vee d$, or $b \leqq c \vee d$, or $a \wedge b \leqq c$, or $a \wedge b \leqq d$. If, on the other hand, $a^{(i)} \wedge b^{(i)} \nleqq c_{(i)} \vee d_{(i)}$

for all $i \in I$ then, by Lemma 1, $a \leqq c \vee d$, or $b \leqq c \vee d$, or $a \wedge b \leqq c$, or $a \wedge b \leqq d$. Thus if each K_i satisfies (F1) so does K.

Now, for each $i \in I$, let K_i satisfy (F2). Let $x, y, z, u \in K$ with $x \vee y = x \vee z = u$. To show that (F2) holds in K we need only show that $u \leqq x \vee (y \wedge z)$.

Write $u = u_0 \vee \cdots \vee u_{n-1}$, $n \geqq 1$, with the u_j satisfying the conditions of Lemma 3. We show that $u_j \leqq x \vee (y \wedge z)$ for each $j < n$. First, let $j < n$ and let $u_j \in L_i$ for some $i \in I$. Then, by Lemma 2, $u_j \leqq u_{(i)}$. But $u_{(i)} = x_{(i)} \vee y_{(i)} = x_{(i)} \vee z_{(i)}$ and so, since K_i satisfies (F2), $u_{(i)} = x_{(i)} \vee (y_{(i)} \wedge z_{(i)}) \leqq x \vee (y \wedge z)$. Thus $u_j \leqq x \vee (y \wedge z)$.

If, on the other hand, $u_j \notin \bigcup (L_i \mid i \in I)$, then there are $a, b \in L$ such that $a > u_j$, $b > u_j$, and $a \wedge b = u_j$ by condition (i) of Lemma 3. Now $a \wedge b \leqq u = x \vee y$. The possibilities $a \leqq x \vee y$ or $b \leqq x \vee y$ contradict condition (iv) of Lemma 3. The possibility $a^{(i)} \wedge b^{(i)} \leqq x_{(i)} \vee y_{(i)}$ contradicts condition (iii) of Lemma 3. Thus, by Lemma 1, $u_j \leqq x$ or $u_j \leqq y$. Similarly $u_j \leqq x$ or $u_j \leqq z$. Thus $u_j \leqq x \vee (y \wedge z)$ whenever $u_j \notin \bigcup (L_i \mid i \in I)$.

We have thus shown that $u_j \leqq x \vee (y \wedge z)$ for all $j < n$, and we conclude that $u \leqq x \vee (y \wedge z)$, establishing that K satisfies (F2).

The situation involving (F3) is just the dual of that involving (F2), and so the proof is complete.

We note that any distributive lattice satisfies (F2) and (F3) and thus a corollary of our result is the observation that a free product of distributive lattices satisfies (F2) and (F3).

A potentially more significant conclusion from the Theorem is based on the observation that the two-element chain satisfies all (Fn). Thus, if K is a sublattice of L with no proper covers, then K satisfies all (Fn). In other words, parts of the free product where the L_i have no direct influence behave exactly like a free lattice.

References

1. G. Grätzer, H. Lakser and C. R. Platt, *Free products of lattices*, Fund. Math. **69** (1970), 233–240. MR **43** #116.

2. F. Galvin and B. Jónsson, *Distributive sublattices of a free lattice*, Canad. J. Math. **13** (1961), 265–272. MR **23** #A78.

3. B. Jónsson, *Sublattices of a free lattice*, Canad. J. Math. **13** (1961), 256–264. MR **23** #A818.

4. ———, *Relatively free products of lattices*, Algebra Universalis **1** (1971/72), 362–373. MR **46** #106.

5. H. Lakser, *Normal and canonical representations in free products of lattices*, Canad. J. Math. **22** (1970), 394–402. MR **42** #144.

6. P. M. Whitman, *Free lattices*, Ann. of Math. (2) **42** (1941), 325–330. MR **2**, 244.

Department of Mathematics, University of Manitoba, Winnipeg, Manitoba Canada