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Z_2-EQUIVARIANT IMMERSIONS AND EMBEDDINGS UP TO COBORDISM

MICHAEL C. BIX

ABSTRACT. The manifolds in an additive basis for the cobordism ring of manifolds with involution are Z_2-equivariantly immersed and embedded into representation spaces having the smallest possible number of nontrivial factors up to cobordism.

Let σ be the Z_2-action on RP^n which is given in homogeneous coordinates by $\sigma[x_0, x_1, \cdots, x_n]=[-x_0, x_1, \cdots, x_n]$. Define $\Gamma: \eta_n^{Z_2} \to \eta_{n+1}^{Z_2}$ by

$$\Gamma[M, T] = \left[\frac{M \times S^1}{(m, z) \sim (Tm, -z)}, [m, z] \mapsto [m, \bar{z}]\right],$$

where $\eta_*^{Z_2}$ is the bordism group of unrestricted involutions. Γ can be defined in the same way for manifolds which are not compact. $\{1\} \cup \{\Gamma^i(RP^{n_1}, \sigma) \times (RP^{n_2}, \sigma) \times \cdots \times (RP^{n_k}, \sigma) \times (M^m, 1) | i \geqq 0, k \geqq 1, n_1 \geqq n_2 \geqq \cdots \geqq n_k > 1, [M]_2 \neq 0\}$, where by Γ^0 we mean the identity, and by $[M]_2$ we mean the class of the closed C^∞ manifold M in the unoriented cobordism ring (Thom [8]), is an additive set of generators for $\eta_*^{Z_2}$ (Alexander [1], Stong [7]). Let $(-1)^r \oplus 1^s$ denote R^{r+s} furnished with the Z_2-action $(x_1, \cdots, x_{r+s}) \to (-x_1, \cdots, -x_r, x_{r+1}, \cdots, x_{r+s})$. For each element (N, S) in the additive basis for $\eta_*^{Z_2}$, we wish to find an integer r_N (resp. r_N') such that there is a manifold with involution in the bordism class of (N, S) which can be Z_2-equivariantly embedded (immersed) in a representation space

$$(-1)^{r_N} \oplus 1^s \qquad ((-1)^{r_N'} \oplus 1^s)$$

for some s and such that no manifold with involution in the bordism class of (N, S) can be Z_2-equivariantly embedded (immersed) in a representation space

$$(-1)^{r_N - 1} \oplus 1^t \qquad ((-1)^{r_N' - 1} \oplus 1^t)$$

for any t. This is a generalization to the case of Z_2-manifolds of the work

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by Brown [2], [3], and Liulevicius [6] on immersions and embeddings up to cobordism.

PROPOSITION 1. *$\Gamma^i(RP^{n_1}, \sigma)\times(RP^{n_2}, \sigma)\times\cdots\times(RP^{n_k}, \sigma)\times(M^m, 1)$ can be Z_2-equivariantly embedded in $(-1)^{i+n_1+n_2+\cdots+n_k}\oplus 1^s$, for some s.*

LEMMA 2. *If $e:(M, T)\to(N, S)$ is a Z_2-equivariant embedding, then so is $\Gamma^i e:\Gamma^i(M, T)\to\Gamma^i(N, S)$, where $\Gamma^i e$ is defined inductively by $\Gamma e[m, z]=[e(m), z]$.*

LEMMA 3. *$\gamma_{r,s,i}:\Gamma^i((-1)^r\oplus 1^s)\to(-1)^{r+i}\oplus 1^{s+ir+i(i+1)/2}$ is a Z_2-equivariant embedding, where $\gamma_{r,s,i}$ is defined inductively on i by*

$$\gamma_{r,s,1}[x_1,\cdots,x_r,x_1',\cdots,x_s',y,z]$$
$$=(x_1z,x_2z,\cdots,x_rz,2yz,x_1y,x_2y,\cdots,x_ry,x_1',x_2',\cdots,x_s',y^2-z^2),$$

where (y, z) are the coordinates of a point in S^1.

PROOF OF PROPOSITION 1. Z_2-equivariant embeddings of (RP^n, σ) into representation spaces can be constructed from techniques of Hopf [5]. A system of n equations, each of which is a real symmetric bilinear form in the variables $x_1, \cdots, x_r$ and $y_1, \cdots, y_r$, whose only real solutions are of the type $x_1=\cdots=x_r=0$ or $y_1=\cdots=y_r=0$, determines an embedding of RP^{r-1} in S^{n-1}. The following $\frac{1}{2}n(n+1)+n+1$ equations:

$$f^1(x_1,\cdots,x_{n+1},y_1,\cdots,y_{n+1}) = x_1y_2 + x_2y_1 = 0,$$
$$f^2(x_1,\cdots,x_{n+1},y_1,\cdots,y_{n+1}) = x_1y_3 + x_3y_1 = 0,$$
$$\vdots$$
$$f^n(x_1,\cdots,x_{n+1},y_1,\cdots,y_{n+1}) = x_1y_{n+1} + x_{n+1}y_1 = 0,$$
$$f^{n+1}(x_1,\cdots,x_{n+1},y_1,\cdots,y_{n+1}) = x_2y_3 + x_3y_2 = 0,$$
$$\vdots$$
$$f^{n(n+1)/2}(x_1,\cdots,x_{n+1},y_1,\cdots,y_{n+1}) = x_ny_{n+1} + x_{n+1}y_n = 0,$$
$$f^{n(n+1)/2+1}(x_1,\cdots,x_{n+1},y_1,\cdots,y_{n+1}) = x_1y_1 = 0,$$
$$\vdots$$
$$f^{n(n+1)/2+n+1}(x_1,\cdots,x_{n+1},y_1,\cdots,y_{n+1}) = x_{n+1}y_{n+1} = 0,$$

define an embedding $e_n: RP^n \to S^{n(n+1)/2+n}$ for $n \geqq 2$, which is given in homogeneous coordinates by

$$e_n[x_0, x_1, \cdots, x_n] = \left(\frac{2x_0x_1}{c}, \frac{2x_0x_2}{c}, \cdots, \frac{2x_0x_n}{c}, \frac{2x_1x_2}{c}, \cdots, \frac{2x_{n-1}x_n}{c}, \frac{x_0^2}{c}, \cdots, \frac{x_n^2}{c}\right),$$

where c is the length of the vector $(2x_0x_1, 2x_0x_2, \cdots, 2x_0x_n, 2x_1x_2, \cdots, 2x_{n-1}x_n, x_0^2, \cdots, x_n^2)$ in $R^{n(n+1)/2+n+1}$. When we consider Z_2-actions on these spaces, the embedding $e_n: (RP^n, \sigma) \to S((-1)^n \oplus 1^{n(n+1)/2+1})$ is a Z_2-equivariant map, where S denotes the associated sphere bundle.

$$P_n = (\underbrace{0, 0, \cdots, 0}_{(n(n+1)/2-1 \text{ times})}, \underbrace{1, 0, \cdots, 0}_{(n+1 \text{ times})})$$

does not lie in the image of e_n. Let $s_n: S^n \to R^n$ denote the stereographic projection map from the point P_n. Then

$$e'_n = s_{n(n+1)/2+n} e_n: (RP^n, \sigma) \to (-1)^n \oplus 1^{n(n+1)/2}$$

is a Z_2-equivariant embedding. Lemmas 2 and 3 then give a Z_2-equivariant embedding

$$e_{i,n}: \Gamma^i(RP^n, \sigma) \to (-1)^{n+i} \oplus 1^{n(n+1)/2+in+i(i+1)/2}.$$

Let $e_M: M \to 1^{2m}$ be an embedding. Then we have a Z_2-equivariant embedding

$$\begin{aligned} &e_{i,n_1} \times e'_{n_2} \times \cdots \times e'_{n_k} \times e_M: \\ &\quad \Gamma^i(RP^{n_1}, \sigma) \times (RP^{n_2}, \sigma) \times \cdots \times (RP^{n_k}, \sigma) \times (M^m, 1) \\ &\qquad \to ((-1)^{i+n_1} \oplus 1^{n_1(n_1+1)/2+in_1+i(i+1)/2}) \times ((-1)^{n_2} \oplus 1^{n_2(n_2+1)/2}) \times \cdots \\ &\qquad\quad \times ((-1)^{n_k} \oplus 1^{n_k(n_k+1)/2}) \times 1^{2m}, \end{aligned}$$

as required in Proposition 1.

PROPOSITION 4. *Let $i: (M, T) \to (N, S)$ be a Z_2-equivariant immersion. If $F(M, T)$, the fixed point set of M under the action of T, has a nonempty component of codimension k in M, then $F(N, S)$ has a nonempty component of codimension $\geqq k$ in N.*

PROOF. Choose a point p in a component F_p of $F(M, T)$ which has codimension k in M. Let $F_{i(p)}$ be the component of $F(N, S)$ which contains $i(p)$. Let $S(\nu_M^k)$ [resp. $S(\nu_N^l)$] be the sphere bundle associated to the normal bundle to F_p [$F_{i(p)}$] in (M, T) [in (N, S)]. The Z_2-action which $S(\nu_M^k)$ inherits from (M, T) is the antipodal action on each fibre. Let S_p^{k-1} [$S_{i(p)}^{l-1}$] be the restriction of the total space of $S(\nu_M^k)$ [$S(\nu_N^l)$]

to those points which lie above p $[i(p)]$ in the fibre bundle. Since $i|_{S_p^{k-1}}: S_p^{k-1} \to S_{i(p)}^{l-1}$ is an immersion, $k-1 \leqq l-1$. So $l \geqq k$.

PROPOSITION 5. *If* $[M_0^{i+n_1+\cdots+n_k+m}, T] \in \eta_{i+n_1+\cdots+n_k+m}^{Z_2}$ *such that*

$$[M_0, T] = [\Gamma^i(RP^{n_1}, \sigma) \times (RP^{n_2}, \sigma) \times \cdots \times (RP^{n_k}, \sigma) \times (M^m, 1)] \in \eta_{i+n_1+\cdots+n_k+m}^{Z_2},$$

and such that there is a Z_2*-equivariant immersion* $f: (M_0, T) \to (-1)^r \oplus 1^s$, *then* $r \geqq i+n_1+n_2+\cdots+n_k$ *and* $s \geqq \min(n_1, n_2, \cdots, n_k)-1+m$.

PROOF. We use the symbol $\cup$ to denote the disjoint union. $F(RP^{n_j}, \sigma) = * \cup RP^{n_j-1}$ and $F(\Gamma(N, S)) = N \cup F(N, S)$. Hence

$$F(\Gamma^i(RP^{n_1}, \sigma)) = * \cup RP^{n_1-1} \cup RP^{n_1} \cup \Gamma^1(RP^{n_1}, \sigma) \cup \cdots \cup \Gamma^{i-1}(RP^{n_1}, \sigma).$$

$$\begin{aligned} &F(\Gamma^i(RP^{n_1}, \sigma) \times (RP^{n_2}, \sigma) \times \cdots \times (RP^{n_k}, \sigma) \times (M^m, 1)) \\ &\quad = F(\Gamma^i(RP^{n_1}, \sigma)) \times F(RP^{n_2}, \sigma) \times \cdots \times F(RP^{n_k}, \sigma) \times F(M^m, 1) \\ &\quad = M \cup (\text{components of dimension} \geqq \min(n_1, n_2, \cdots, n_k) - 1 + m). \end{aligned}$$

But the unoriented cobordism class of the fixed point set is an invariant of the cobordism class of a manifold with involution. Thus $F(M_0, T) =$ (components of dimension m, at least one of which is nonempty) $\cup$ (components of dimension $\geqq \min(n_1, n_2, \cdots, n_k)-1+m$), since $[M]_2 \neq 0$. Therefore $r \geqq i+n_1+n_2+\cdots+n_k$ by Proposition 4, and $F(M_0, T)$ must contain at least one nonempty component of dimension $\geqq \min(n_1, n_2, \cdots, n_k)-1+m$, because $[M]_2 \neq 0$ (Conner and Floyd [**4**]). Since f is an immersion, $s \geqq \min(n_1, n_2, \cdots, n_k)-1+m$.

Combining Propositions 1 and 5 we have the following result for manifolds which form an additive basis for $\eta_*^{Z_2}$:

THEOREM 6. $\Gamma^i(RP^{n_1}, \sigma) \times (RP^{n_2}, \sigma) \times \cdots \times (RP^{n_k}, \sigma) \times (M^m, 1)$ *can be* Z_2*-equivariantly embedded in* $(-1)^{i+n_1+n_2+\cdots+n_k} \oplus 1^s$ *for some* s, *and cannot be* Z_2*-equivariantly immersed up to cobordism in* $(-1)^{i+n_1+n_2+\cdots+n_k-1} \oplus 1^t$ *for any* t, *where* $i \geqq 0$, $k \geqq 1$, $n_1 \geqq n_2 \geqq \cdots \geqq n_k > 1$, $[M^m]_2 \neq 0$ *in* η_m.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637