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# CHARACTERIZATION OF ABSTRACT COMPOSITION OPERATORS

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ABSTRACT. A *composition operator* on $L^p(X, \mu)$ is (roughly) an operator $T$ induced by a point transformation $\phi$ on $X$ by $Tf = f \cdot \phi$.

Characterizations are given of abstract Hilbert-space operators which can be represented (via unitary equivalence) as composition operators. Representation on $L^2(J, m)$ ($J$ an interval of the real line, $m$ a Borel measure) and on $L^2(0, 1)$ (Lebesgue measure) are considered.

Also, any bounded measure-algebra transformation which preserves disjoint unions is a sigma-homomorphism.

*Notation.* Throughout this paper, $H$ denotes a Hilbert space of countably infinite dimension, and $T$ denotes a bounded linear operator on $H$.

**Definition.** An operator $T$ on $L^p(X, \mu)$ induced by a transformation $\phi$: $X_1 \to X$, $X_1 \subset X$, by the formula $Tf = (f \cdot \phi)\chi_{X_1}$ is called a *composition operator*.

**Definition** (Choksi [2, p. 89]). A subset $C$ of $H$ is a *C-family* if it satisfies the following conditions:

(1) $C$ is closed (topologically) in $H$.

(2) $C$ is total in $H$.

(3) $C$ has a unique element $i$ of maximal norm $\|i\| = 1$.

(4) If $f \in C$, then $i - f \in C$ and $(f, i - f) = 0$.

(5) If $f, g, h \in C$ and $(h, i - g) = (g, f) = 0$, then $(h, f) = 0$.

(6) If $f, g \in C$, then $f + g \in C$ if and only if $(f, g) = 0$.

(7) If $f \in C$ and $f \neq i$, then $C$ has an element $g \neq 0$ such that $g \neq i - f$ and $(f, g) = 0$.

(8) If $f, g \in C$, then $C$ has a unique element $h$ such that $(h, i - f) = (h, i - g) = 0$ and $\|h\|^2 = (f, g)$. $(f, g)$ is real and nonnegative.

**Theorem** (Choksi [2, p. 94]). *A subset $C$ of $H$ corresponds to the char-*

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*acteristic functions under some isometry of $H$ with $L^2(0, 1)$, Lebesgue measure, if and only if it is a $C$-family.*

**Definition** (Sz.-Nagy). A subset $C$ of $H$ is an *$N$-family* if it satisfies the following conditions. We say $f < g$ if $(f, g) = \|f\|^2$.

(1) $C$ is total in $H$.

(2) If $f, g \in C$, then $f - g \in C$ if and only if $g < f$.

(3) For all $f, g$ in $C$, there are elements $h, k$ of $C$ such that $f + g = h + k$, $h < f$, $h < g$.

(4) If $(f_k)$ is an increasing sequence from $C$ with a limit $f$ in $H$, then $f$ is in $C$.

**Theorem** (Sz.-Nagy). *A subset $C$ of $H$ corresponds to the characteristic functions under some isometry of $H$ with $L^2(J, m)$, $J$ a (finite or infinite) interval on the real line, $m$ a Borel measure, if and only if $C$ is an $N$-family*

**Definition.** Let $\Phi$ be a transformation from a measure algebra $(A, \mu)$ to another, $(B, \nu)$. The *norm* of $\Phi$ is

$$\|\Phi\| = \sup\{\nu[\Phi(\alpha)]/\mu(\alpha):\ 0 < \mu(\alpha) < \infty,\ \alpha \in A\}.$$

$\Phi$ is *bounded* if $\|\Phi\| < \infty$. $\Phi$ is *additive* if $\alpha \wedge \beta = 0$ implies $\Phi(\alpha \vee \beta) = \Phi(\alpha) \vee \Phi(\beta)$ and $\Phi(\alpha) \wedge \Phi(\beta) = 0$.

**Lemma.** *Every bounded additive transformation from a totally finite measure algebra into an arbitrary measure algebra is a sigma-homomorphism.*

**Proof.** Let $\Phi: (A, \mu) \to (B, \nu)$ be bounded and additive, $(A, \mu)$ totally finite. For $\alpha, \beta$ in $A$,

$$\Phi(\alpha) = \Phi(\alpha \wedge \beta) \vee \Phi(\alpha - \beta) \quad \text{and} \quad \Phi(\beta) = \Phi(\alpha \wedge \beta) \vee \Phi(\beta - \alpha),$$

both disjoint unions, by additivity. Taking intersections on both sides, we see that $\Phi$ preserves (by induction) finite intersections. This, together with additivity, implies that $\Phi$ preserves differences, for

$$\Phi(\alpha - \beta) = \Phi(\alpha) - \Phi(\alpha \wedge \beta) = \Phi(\alpha) - [\Phi(\alpha) \wedge \Phi(\beta)] = \Phi(\alpha) - \Phi(\beta).$$

Suppose $\alpha =$ (countable) $\bigvee \alpha_K$ with $\alpha, \alpha_K$ in $A$. Since additivity implies monotonicity,

$$\bigvee \Phi(\alpha_K) \leq \Phi(\alpha).$$

If they are not equal, then $\nu[\Phi(\alpha) - \bigvee \Phi(\alpha_K)] = d > 0$.
For some integer $s$, $\mu(\alpha - \bigvee_{K=1}^{s} \alpha_K) < d/\|\Phi\|$.

Then

$$\|\Phi\| > \frac{\nu \cdot \Phi(\alpha - \bigvee_{K=1}^{s} \alpha_K)}{\mu(\alpha - \bigvee_{K=1}^{s} \alpha_K)} > \frac{\nu \cdot \Phi(\alpha - \bigvee \alpha_K)}{\mu(\alpha - \bigvee_{K=1}^{s} \alpha_K)} > \|\Phi\|,$$

a contradiction. So $\Phi$ preserves countable unions, and is hence a sigma-homomorphism.

**Theorem 1.** *An operator $T$ on $H$ is unitarily equivalent to a composition operator on $L^2(0, 1)$, Lebesgue measure, if and only if there is a $C$-family $C$ in $H$ such that $TC \subset C$.*

**Proof.** The characteristic functions on $(0, 1)$ do form a $C$-family, and every composition operator takes characteristic functions into characteristic functions.

Conversely, suppose $TC \subset C$ where $C$ is a $C$-family. By Choksi's theorem we may (by unitary equivalence) identify $H$ with $L^2(0, 1)$, Lebesgue measure, and $C$ with the characteristic functions, with $TC \subset C$. If $(A, \mu)$ is the measure algebra (Lebesgue) of $(0, 1)$ and $\alpha \in A$, then $T\chi_\alpha$ is the characteristic function of an element $\Phi(\alpha)$ of $A$. This defines a transformation $\Phi$ on $A$.

If $\alpha \wedge \beta = 0$, then

$$\chi_{\Phi(\alpha \vee \beta)} = T\chi_{\alpha \vee \beta} = T\chi_\alpha + T\chi_\beta = \chi_{\Phi(\alpha)} + \chi_{\Phi(\beta)}$$

so

$$\Phi(\alpha \vee \beta) = \Phi(\alpha) \vee \Phi(\beta),$$

disjoint union, and so $\Phi$ is additive. Also

$$\mu[\Phi(\alpha)] = \|\chi_{\Phi(\alpha)}\|^2 = \|T\chi_\alpha\|^2 \le \|T\|^2 \mu(\alpha)$$

so

$$\|\Phi\| \le \|T\|^2$$

and $\Phi$ is bounded. By the above Lemma, $\Phi$ is therefore a sigma-homomorphism, and by [6] there is a measurable point transformation $\phi$: $X_1 \to (0, 1)$ such that $\phi^{-1}E \in \Phi(\alpha)$ for all Borel sets $E$, where $\alpha$ is the equivalence class of $E$ in $A$, and $X_1$ is a measurable subset of $(0, 1)$.

Let $T_\phi$ be the composition operator on $L^2(0, 1)$ induced by $\phi$, $E$ any Borel set, and $\alpha$ its equivalence class in $A$. Then

$$T\chi_\alpha = \chi_{\Phi(\alpha)} = [\chi_{\phi^{-1}E}] = [\chi_E \circ \phi] = T_\phi \chi_\alpha$$

where $[\circ]$ denotes equivalence class in $L^2$. Since $T_\phi$ is bounded by its bound on characteristic functions (a routine exercise, or see [4] or [5]), $T$ and $T_\phi$ are bounded linear operators which agree on characteristic functions, and hence $T = T_\phi$.

**Theorem 2.** *An operator $T$ on $H$ is unitarily equivalent to a composition operator if and only if there is an $N$-family $C$ in $H$ such that $TC \subset C$.*

**Proof.** Characteristic functions on a measure space do form an $N$-family, which again is always invariant under a composition operator.

If $T$ leaves an $N$-family invariant, then by Sz.-Nagy's theorem $T$ is unitarily equivalent to an operator $S$ on $L^2(J, m)$ ($J$ an interval, $m$ a Borel measure) which takes characteristic functions into characteristic functions. As in Theorem 1, $S\chi_\alpha = \chi_{\Phi(\alpha)}$ where $\Phi$ is a sigma-homomorphism on the collection of sets having finite measure. (Restrict $\Phi$ to any finite subinterval and apply the above Lemma.) By the first corollary to Sikorski's theorem [6], there is a measurable transformation $\phi: J_0 \to J$ with $\phi^{-1}E \in \Phi(\alpha)$ for all Borel sets $E$, $\alpha$ being the equivalence class of $E$ in the measure algebra of $J$, and $J_0$ being a measurable subset of $J$. Then $\phi$ induces $S$ as in the proof of Theorem 1.

**Corollary.** *Any composition operator on a separable infinite-dimensional Hilbert space can be represented on an interval of the real line with a Borel measure.*

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