PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 3, September 1974

# PRODUCTS OF $M$-SPACES

C. BANDY[1]

ABSTRACT. The author associates with each pair $X$, $Y$ of $M$-spaces such that $X \times Y$ is not an $M$-space, a pair of countably compact closed subspaces $A \subset X$, $B \subset Y$ such that $A \times B$ is not countably compact, and for each pair $A$, $B$ of countably compact spaces whose product is not countably compact, there is a pair of $M$-spaces $S$, $T$ (in fact, $S$ and $T$ are countably compact) such that $S \times T$ is not an $M$-space and such that $A$ and $B$ are closed subspaces of $S$ and $T$ respectively.

During the last few years $M$-spaces have become of interest to people in general topology. $M$-spaces are topological spaces having a normal sequence satisfying certain properties. For the interest of the reader we list two ways in which normal sequences have been used.

(1) A necessary and sufficient condition for metrizability of a $T_1$-space is the existence of a normal sequence $\{U_n \mid n \in N\}$ such that $\bigcup\{U_n \mid n \in N\}$ form a base for the topology.

(2) If $X$ is a completely regular space, a necessary and sufficient condition that an open cover $U$ of $X$ be part of some uniformity on $X$ is that $U$ be a member of some normal sequence for $X$.

Throughout, all spaces are assumed to be Hausdorff. The set of positive integers will be denoted by $N$. If $U$ is an open covering of a topological space $X$, then the union of all members of $U$ containing the point $x$ of $X$ will be denoted by $st(x, U)$. The sequence $\{U_n \mid n \in N\}$ is a *normal sequence* for $X$ provided $U_n$ is an open cover of $X$ and $U_{n+1}$ star-refines $U_n$. The sequence $\{U_n | n \in N\}$ is an *$M$-sequence* for $X$ provided that if $x$ is a point of $X$ and $x_n$ is in $st(x, U_n)$ for each $n$ in $N$, then $\{x_n | n \in N\}$ has a cluster point. A topological space $X$ is called an *$M$-space* (see [2]) provided $X$ has a normal sequence that is also an $M$-sequence.

Thus $M$-spaces are natural generalizations of metric and countably compact spaces.

Received by the editors June 6, 1973 and, in revised form, September 30, 1973.
*AMS (MOS) subject classifications* (1970). Primary 54E99; Secondary 54B10.
*Key words and phrases.* $M$-space, normal sequence, $M$-sequence.

[1]This paper represents a section of the author's thesis written at the University of Houston under the direction of Professor J. N. Younglove.

In [2] it is proved that a topological space is an $M$-space if, and only if, it is the inverse image of a metric space under a quasiperfect map (continuous, closed, surjective, and the inverse image of each point is countably compact). This map gives a natural decomposition of an $M$-space into closed, pairwise disjoint, countably compact sets. Lemma 1 presents a proof of this decomposition since the construction is used later in the paper.

I would also like to point out that a topological space is a paracompact $M$-space if, and only if, it is the inverse image of a metric space under a perfect map; and paracompact $M$-spaces and paracompact $p$-spaces (in the sense of Arhangel'skiĭ) are equivalent.

**Lemma 1.** *Let* $\{U_n | n \in N\}$ *be a normal sequence for the* $M$*-space* $X$*. Then* $\{\bigcap_n st(x, U_n) | x \in X\}$ *is a decomposition of* $X$ *into closed, pairwise disjoint, countably compact sets.*

For each $x$ in $X$, the set $\bigcap_n st(x, U_n)$ is closed since $st^2(x, U_{n+1})$ is contained in $st(x, U_n)$ so the closure of $st(x, U_{n+1})$ is contained in $st(x, U_n)$. Hence $\bigcap_n st(x, U_n)$ equals $\bigcap_n (\mathrm{cl}\, st(x, U_n))$. The set $\bigcap_n st(x, U_n)$ is countably compact because if $\{x_n | n \in N\}$ is a sequence of points in $\bigcap_n st(x, U_n)$ then $x_n$ is in $st(x, U_n)$ and since $\{U_n | n \in N\}$ is an $M$-sequence, $\{x_n | n \in N\}$ has a cluster point. To show the set $\{\bigcap_n st(x, U_n) | x \in X\}$ is pairwise disjoint suppose the point $y$ is not in $\bigcap_n st(x, U_n)$. This implies that there is a positive integer $m$ so that $y$ is not in $st(x, U_m)$. Therefore, $y$ is not in $st^2(x, U_{m+1})$, implying that there is no point common to both $st(x, U_{m+1})$ and $st(y, U_{m+1})$.

From the last step in the above proof, it also follows that any cluster point of a sequence $\{x_n | n \in N\}$, with $x_n$ in $st(x, U_n)$, is in $\bigcap_n st(x, U_n)$.

**Theorem 1.** *If* $X$ *and* $Y$ *are* $M$*-spaces such that* $X \times Y$ *is not an* $M$*-space, then there is a countably compact closed subspace* $A$ *contained in* $X$ *and a countably compact closed subspace* $B$ *contained in* $Y$ *such that* $A \times B$ *is not countably compact.*

Let $X$ and $Y$ be $M$-spaces such that $X \times Y$ is not an $M$-space and let $\{U_n | n \in N\}$ and $\{V_n | n \in N\}$ be normal $M$-sequences for $X$ and $Y$ respectively. The set

$$\{W_n | W_n = \{U \times V | U \in U_n \text{ and } V \in V_n \text{ for each } n \text{ in } N\}\}$$

is a normal sequence for $X \times Y$, because if $(x, y)$ is a point of $X \times Y$ then $\{U | U \in U_n$ for $n > 1$ and $x$ in $U\}$ is contained in some open subset $u$ of $U_{n-1}$

and $\{V | V \in V_n$ for $n > 1$ and $y$ in $V\}$ is contained in some open subset $v$ of $V_{n-1}$, therefore $\{U \times V | x \in U \in U_n$ and $y \in V \in V_n\}$ is contained in $u \times v$.

Since $X \times Y$ is not an $M$-space there is a point $(x, y)$ in $X \times Y$ and an infinite closed discrete set $\{(x_n, y_n) | n \in N\}$ contained in $X \times Y$ such that $(x_n, y_n)$ is in $st((x, y), W_n)$. The set

$$A = \{x_n | n \in N\} \cup \left[\bigcap_n st(x, U_n)\right]$$

is countably compact since $\bigcap_n st(x, U_n)$ is countably compact by Lemma 1 and every subsequence of $\{x_n | n \in N\}$ has a cluster point in $\bigcap_n st(x, U_n)$. Similarly

$$B = \{y_n | n \in N\} \cup \left[\bigcap_n st(y, U_n)\right]$$

is countably compact but the product of $A$ and $B$ is not countably compact since $A \times B$ contains the infinite closed discrete set $\{(x_n, y_n) | n \in N\}$.

**Corollary 1.** *If $X$ and $Y$ are normal $M$-spaces such that $X \times Y$ is not an $M$-space, then there is a pair of normal countably compact closed subspaces $A \subset X$, $B \subset Y$ such that $A \times B$ is not countably compact.*

**Corollary 2** [2, Theorem 6.4]. *The product of a pair of paracompact $M$-spaces is an $M$-space.*

In [3] J. Novák proved the existence of two countably compact subsets $A$ and $B$ of $\beta(N)$, the Stone-Čech compactification of the positive integers, whose product is not countably compact. A. K. Steiner [4] used a modification of Novák's example to construct two countably compact spaces $X$ and $Y$ such that $X \times Y$ is an $M$-space, but is not countably compact. Theorem 2 also uses Novák's example. Although it has already been proved that the product of $M$-spaces need not be an $M$-space [1] the following example is simpler and is necessary in the corollary to follow.

**Theorem 2.** *If $A$ and $B$ are countably compact spaces whose product is not countably compact then there is a pair of $M$-spaces $S$ and $T$ (in fact $S$ and $T$ are countably compact) such that $S \times T$ is not an $M$-space and $A$ and $B$ are closed subspaces of $S$ and $T$ respectively.*

Let $[1, \Omega]$ be the space of countable ordinals together with the first uncountable ordinal $\Omega$ and having the order topology. Define a topological space $S$ by replacing each nonlimit ordinal in $[1, \Omega]$ with a homeomorphic copy of $A$. An open set containing a point of $S$ will be an open subset of the homeomorphic copy of $A$ containing the point, if the point lies in a homeomorphic copy of $A$. If the point $\lambda$ of $S$ is a limit ordinal of $[1, \Omega]$ then an open set containing $\lambda$ is $\lambda$ together with the union of all homeomorphic copies of $A$ replacing the ordinals $\alpha$ through $\lambda$ in $[1, \Omega]$ for some $\alpha$ preceding $\lambda$ together with the union of all limit ordinals of $[1, \Omega]$ between $\alpha$ and $\lambda$. Define a topological space $T$ similarly but using sets homeomorphic to $B$. Then each of $S$ and $T$ is countably compact since each sequence of points in $S$ has the property that either infinitely many members of the sequence are in some homeomorphic copy of $A$, in which case the sequence has a cluster point, or each homeomorphic copy of $A$ contains at most finitely many members of the sequence, in which case the least limit ordinal greater than infinitely many members of the sequence is a cluster point of the sequence. (We say $\lambda$ is greater than $\alpha$ in $S$ provided $\lambda$ and $\alpha$ are limit ordinals of $[1, \Omega]$ and $\lambda$ is greater than $\alpha$ in $[1, \Omega]$. The limit ordinal $\lambda$ of $[1, \Omega]$ is greater than each point of a homeomorphic copy of $A$ replacing an ordinal of $[1, \Omega]$ less than $\lambda$, and each point of a homeomorphic copy of $A$ replacing an ordinal $\alpha$ of $[1, \Omega]$ is less than each point of a homeomorphic copy of $A$ replacing an ordinal $\beta$ greater than $\alpha$.) Now $S$ and $T$ are $M$-spaces but $S \times T$ is not an $M$-space since $\bigcap_n st((\Omega, \Omega), W_n)$, for any sequence of open covers $\{W_n \mid n \in N\}$ of $S \times T$, contains a homeomorphic copy of $A \times B$. This follows since the intersection of countably many open subsets of $S$, each containing $\Omega$, contains a homeomorphic copy of $A$ and the intersection of countably many open subsets of $T$, each containing $\Omega$, contains a homeomorphic copy of $B$; hence the set $\bigcap_n st((\Omega, \Omega), W_n)$ contains a homeomorphic copy of $A \times B$.

**Corollary 3.** *If $A$ and $B$ are two normal countably compact spaces whose product is not countably compact then there are two normal $M$-spaces $S$ and $T$ whose product is not an $M$-space.*

*Question* 1. Is the product of two normal $M$-spaces an $M$-space?

*Question* 2. Are there two normal countably compact spaces whose product is not countably compact?

The author would like to thank the referee for many helpful suggestions.

## REFERENCES

1. T. Isiwata, *The product of M-spaces need not be an M-space*, Proc. Japan Acad. 45 (1969), 154–156. MR 39 #6242.

2. K. Morita, *Products of normal spaces with metric spaces*, Math. Ann. 154 (1964), 365–382. MR 29 #2773.

3. J. Novák, *On the cartesian product of two compact spaces*, Fund. Math. 40 (1953), 106–112. MR 15, 640.

4. A. K. Steiner, *On the topological completion of M-space products*, Proc. Amer. Math. Soc. 29 (1971), 617–620. MR 43 #8051.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77004

*Current address*: University of Houston Victoria Centre, Victoria, Texas 77901