PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 3, September 1974

THE RIESZ SUMMABILITY OF LOGARITHMIC TYPE

B. KWEE[1]

ABSTRACT. The series $\sum_{n=1}^{\infty} a_n$ is said to be summable (L) to s if $(\log(1-x))^{-1}\sum_{n=1}^{\infty} s_n x^{n+1}/n$, where $s_n = \sum_{\nu=1}^{n} a_\nu$, converges for $0 \le x < 1$ and tends to s when $x \to 1-$. The aim of this paper is to discuss the relation between summability (L) and Riesz summability $(R, \log n, \kappa)$. It is proved that $(R, \log n, \kappa) \subseteq (L)$ holds for $0 \le \kappa \le 1$ and is false for $\kappa > 1$. It is also proved that if $\sum_{n=1}^{\infty} a_n = s(L)$ and bounded $(R, \log n, \kappa)$ for $\kappa \ge 0$ then $\sum_{n=1}^{\infty} a_n = s(R, \log n, \kappa + \delta)$ for every $\delta > 0$.

1. Introduction. Let $\kappa \geq 0$, and let

$$A^{\kappa}(u) = \sum_{\log n < u} (u - \log n)^{\kappa} a_n.$$

If $C^{\kappa}(u) = u^{-\kappa} A^{\kappa}(u) \to s$ as $u \to \infty$, we say the series $\sum_{n=1}^{\infty} a_n$ is summable $(R, \log n, \kappa)$ to s and write $\sum_{n=1}^{\infty} a_n = s(R, \log n, \kappa)$.

If

$$\frac{-1}{\log(1-x)} \sum_{n=1}^{\infty} \frac{s_n x^{n+1}}{n},$$

where $s_n = \sum_{\nu=1}^{n} a_\nu$, converges for $0 \leq x < 1$ and tends to s as $x \to 1-$, we say the series $\sum_{n=1}^{\infty} a_n$ is summable (L) to s.

The relation between summability $(R, \log n, \kappa)$ and (L) will be discussed in this paper. We shall prove

Theorem 1. *The inclusion* $(R, \log n, \kappa) \subseteq (L)$ *holds for* $0 \leq \kappa \leq 1$ *and is false for* $\kappa > 1$.

Theorem 2. *If* $\sum_{n=1}^{\infty} a_n$ *is summable* (L) *and bounded* $(R, \log n, \kappa)$, *then it is summable* $(R, \log n, \kappa + \delta)$ *to the same sum for every* $\delta > 0$.

2. Proof of Theorem 1. For $0 \leq \kappa \leq 1$, $(R, \log n, \kappa) \subseteq (R, \log n, 1)$. But summability $(R, \log n, 1)$ is equivalent to summability (l) defined by

Received by the editors November 1, 1971.

AMS (MOS) subject classifications (1970). Primary 40G99.

[1] The research reported herein has been sponsored in part by the U. S. Army Research and Development Group (Far East).

$$\frac{1}{\log(n+1)} \sum_{\nu=1}^{n} \frac{s_\nu}{\nu} \to s$$

(see [4, Theorem 37]), and, by [4, Theorem 57], $(l) \subseteq (L)$. Hence $(R, \log n, \kappa) \subseteq (L)$ for $0 \leq \kappa \leq 1$.

To prove the second part of the theorem, let $\Sigma_{n=1}^{\infty} b_n$ be the series whose partial sums B_n are defined by

$$t_n = \sum_{\nu=1}^{n} \frac{B_\nu}{\nu} = \gamma_n n^{-it},$$

where $\gamma_n = \Sigma_{\nu=1}^{n} 1/\nu$ and $t \neq 0$. We have

$$(1) \quad \frac{-1}{\log(1-x)} \sum_{n=1}^{\infty} \frac{B_n x^n}{n} = \frac{-(1-x)}{\log(1-x)} \sum_{n=1}^{\infty} t_n x^n = \frac{-(1-x)}{\log(1-x)} \sum_{n=1}^{\infty} \gamma_n n^{-it} x^n.$$

It is easy to verify that $1/\gamma_n$ is totally monotonic. Hence there exists a monotonic increasing function $\chi(x)$ such that $1 = \gamma_n \int_0^1 x^n d\chi(x)$ for $n \geq 0$. It follows from a theorem of Borwein [1] that, if the right-hand side of (1) tends to a finite limit s, then $n^{-it} \to s(A)$. Since $(n+1)^{-it} - n^{-it} = O(n^{-1})$, $n^{-it} \to s$ by a Tauberian theorem for Abel summability, which is impossible. Hence $\Sigma_{n=1}^{\infty} b_n$ is not summable (L).

Let $u = \log w$, $n < w \leq n + 1$. Then

$$\sum_{\nu < w} b_\nu \log \frac{w}{\nu} = \sum_{\nu=1}^{n-1} B_\nu \log \frac{\nu+1}{\nu} + B_n \log \frac{w}{n}$$

$$= \sum_{\nu=1}^{n-1} \frac{B_\nu}{\nu} + O(\log n) = O(u).$$

Hence $\Sigma_{n=1}^{\infty} b_n$ is bounded $(R, \log n, 1)$.

We have

$$\frac{1}{2} \sum_{\mu=1}^{N} \log \frac{\mu+2}{\mu} \sum_{\nu=1}^{\mu} \left(\log \frac{\nu+1}{\nu}\right) B_\nu$$

$$= \sum_{\mu=1}^{N} \frac{1}{\mu} \sum_{\nu=1}^{\mu} \frac{B_\nu}{\nu} + O(1) = \sum_{\mu=1}^{N} \gamma_\mu \mu^{-1-it} + O(1).$$

Let

$$\sigma_n = \sum_{\nu=1}^{n} \nu^{-1-it} = \frac{n^{-it}}{it} + D(t) + o(1).$$

(See [4, p. 333].) Then

$$\sum_{\mu=1}^{N} \gamma_\mu \mu^{-1-it} = -\sum_{\mu=1}^{N} \frac{\sigma_\mu}{\mu+1} + \gamma_{N+1}\sigma_N = o(\log^2 N).$$

It follows from a result of Jurkat [5] that $\sum_{n=1}^{\infty} b_n = O(R, \log n, 2)$. Hence, by the convexity theorem for Riesz summability [3, p. 19] that $\sum_{n=1}^{\infty} b_n = O(R, \log n, \kappa)$ for $\kappa > 1$.

3. Proof of Theorem 2. We shall use the following lemmas.

Lemma 1. *Let k be a nonnegative integer. If $C^k(u)$ is bounded, then, for $t > 0$, the series*

$$\sum_{n=1}^{\infty} a_n n^{-t} \tag{2}$$

is summable (C, k) to

$$\frac{t^{k+1}}{\Gamma(k+1)} \int_0^\infty e^{-tu} A^k(u)\, du. \tag{3}$$

Since summability (C, k) is equivalent to (R, n, k), this is a special case of [3, Theorem 3.51]. Note that there is no need to suppose that k is an integer; but, as it is much easier to prove the result in this case, and as this case is enough for our application, we state the result for this case only.

Lemma 2. *Suppose that, for some $k \geq 0$, $t > 0$, the series* (2) *is summable (C, k). Then the series*

$$\sum_{n=1}^{\infty} s_n (n^{-t} - (n+1)^{-t}) \tag{4}$$

is summable (C, k) to the same sum as (2).

If $k = 0$, we are given that (2) converges. It follows easily that $s_n = o(n^t)$. Hence, by partial summation, (4) converges to the same sum as (2).

Suppose now that $k > 0$. It follows from a theorem of [2] that $\sum_{n=1}^{\infty} s_{n-1} n^{-t-1}$ is summable $(C, k-1)$ to some sum. Hence the sequence $\{s_{n-1} n^{-t}\}$ is summable (C, k) to 0. By the translativity of (C, k), the sequence $\{s_n (n+1)^{-t}\}$ is summable (C, k) to 0. But

$$\sum_{\nu=1}^{n} a_\nu \nu^{-t} = \sum_{\nu=1}^{n} s_\nu (\nu^{-t} - (\nu+1)^{-t}) + s_n (n+1)^{-t},$$

i.e. the nth partial sums of (2) and (4) differ by $s_n(n+1)^{-t}$. Hence the result.

Lemma 3. *Suppose that*

(5) $$f(u) = \sum_{n=1}^{\infty} s_n n^{-1} e^{-nu}$$

converges for all $u > 0$; *suppose that*

(6) $$f(u) = o(u^{-\alpha})$$

as $u \to 0+$ *for every fixed* $\alpha > 0$. *Then, for all* $t > 0$,

(7) $$\sum_{n=1}^{\infty} s_n n^{-t-1}$$

is Abel summable to $(1/\Gamma(t)) \int_0^\infty u^{t-1} f(u)\, du$.

Take $t > 0$ as fixed. Then for any $x > 0$

$$\sum_{n=1}^{\infty} s_n n^{-t-1} e^{-nx} = \frac{1}{\Gamma(t)} \sum_{n=1}^{\infty} s_n n^{-1} \int_0^\infty e^{-n(u+x)} u^{t-1}\, du.$$

Since (5) converges for all $u > 0$, it converges absolutely for all $u > 0$. Applying this result with u replaced by x, we see that the inversion in the order of integration is justified by absolute convergence. Hence

$$\sum_{n=1}^{\infty} s_n n^{-t-1} e^{-nx} = \frac{1}{\Gamma(t)} \int_0^\infty u^{t-1} f(u+x)\, du.$$

Hence it is enough to prove that

(8) $$\int_0^\infty u^{t-1} f(u+x)\, du - \int_0^\infty u^{t-1} f(u)\, du \to 0$$

as $x \to 0+$. Applying (6) with some α satisfying $0 < \alpha < t$, the difference on the left of (8) is

$$o\left(x^{-\alpha} \int_0^x u^{t-1}\, du\right) + o\left(\int_0^{2x} u^{t-1-\alpha}\, du\right) + \int_{2x}^\infty ((u-x)^{t-1} - u^{t-1}) f(u)\, du.$$

The first two terms clearly tend to 0 as $x \to 0+$. The third is

(9) $$O\left(x \int_{2x}^\infty u^{t-2} |f(u)|\, du\right).$$

Again using (6), and using also the result that, for large u, $f(u) = O(e^{-u})$, we find that the expression (9) tends to 0 as $x \to 0+$; hence the lemma.

We can now prove Theorem 2. Let k be an integer with $k \geq \kappa$. Since $\sum_{n=1}^\infty a_n$ is bounded $(R, \log n, \kappa)$, it is also bounded $(R, \log n, k)$. By Lemma 1, the series (2) is summable (C, k) to the expression (3). Hence, by Lemma 2, the series (4) is summable (C, k), and hence Abel summable to the same sum. Now, by the analogue for Riesz bounded series of the limitation theorem for Riesz summable series

$$s_n = O(n^k(\log n)^k). \tag{10}$$

For $0 < t < 1$, we have

$$\frac{1}{n^t} - \frac{1}{(n+1)^t} = t\left(\frac{1}{n^{t+1}} + \sum_{\rho=1}^{k} \frac{c_\rho(t)}{n^{t+\rho+1}} + R_n(t)\right),$$

where

$$R_n(t) = O\left(\frac{1}{n^{t+k+2}}\right) = O\left(\frac{1}{n^{k+2}}\right)$$

uniformly in t, and where $c_\rho(t)$ is, for each ρ, a bounded function of t. Hence, by (10), $\Sigma_{n=1}^{\infty} s_n R_n(t)$ converges uniformly in t. It follows from Lemma 3 that the series (4) is Abel summable to

$$\frac{t}{\Gamma(t)} \int_0^\infty u^{t-1} f(u)\, du + t \sum_{\rho=1}^{k} \frac{c_\rho(t)}{\Gamma(t+\rho)} \int_0^\infty u^{t+\rho-1} f(u)\, du + t \sum_{n=1}^{\infty} s_n R_n(t). \tag{11}$$

Hence the expressions (3) and (11) are equal.

Now for large u, $f(u) = O(e^{-u})$ so that the contribution of the range $u \geq 1$ to the first term in (11) tends to 0. We have $f(u) \sim s \log(1/u)$ as $u \to 0+$,

$$\int_0^1 u^{t-1} \log \frac{1}{u}\, du = \int_0^\infty v e^{-vt}\, dv = t^{-2},$$

and, for fixed $\eta > 0$, $\int_\eta^1 u^{t-1} \log(1/u)\, du = O(1)$ as $t \to 0+$. Hence $(t/\Gamma(t))\int_0^1 u^{t-1} f(u)\, du \to s$ as $t \to 0+$. Since, for $\rho \geq 1$, $\int_0^\infty u^{t+\rho-1} f(u)\, du$ is bounded, and by uniform convergence $\Sigma_{n=1}^{\infty} s_n R_n(t)$ is also bounded Hence the expression (3) tends to s as $t \to 0+$. In other words

$$\frac{t^{k+1}}{\Gamma(k+1)} \int_0^\infty e^{-tu} u^k C^k(u)\, du \to s$$

as $t \to 0+$. Since $C^k(u)$ is bounded, it follows from a theorem of Wiener (as stated, for example, in [4, Theorem 232]) that

$$C^{k+1}(w) = \frac{k+1}{w^{k+1}} \int_0^w u^k C^k(u)\, du \to s$$

as $w \to \infty$. In other words, $\Sigma_{n=1}^{\infty} a_n$ is summable $(R, \log n, k+1)$ to s. The result now follows from the convexity theorem for Riesz summability.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MALAYA, KUALA LUMPUR, MALAYSIA