PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 3, September 1974

RIESZ SEMINORMS WITH FATOU PROPERTIES

C. D. ALIPRANTIS

ABSTRACT. A seminormed Riesz space L_ρ satisfies the σ-Fatou property (resp. the Fatou property) if $\theta \leq u_n \uparrow u$ in L (resp. $\theta \leq u_\alpha \uparrow u$ in L) implies $\rho(u_n) \uparrow \rho(u)$ (resp. $\rho(u_\alpha) \uparrow \rho(u)$). The following results are proved:

(i) A normed Riesz space L_ρ satisfies the σ-Fatou property if, and only if, its norm completion does and L_ρ has (A, 0).

(ii) The quotient space L_ρ/I_ρ has the Fatou property if L_ρ is Archimedean with the Fatou property. ($I_\rho = \{u \in L: \rho(u) = 0\}$.)

(iii) If L_ρ is almost σ-Dedekind complete with the σ-Fatou property, then L_ρ/I_ρ has the σ-Fatou property.

A counterexample shows that (iii) may be false for Archimedean Riesz spaces.

1. **Riesz seminorms.** For notation and terminology not explained below we refer the reader to [5]. A seminormed Riesz space L_ρ is a Riesz space L equipped with a seminorm ρ satisfying $\rho(u) \leq \rho(v)$ whenever $|u| \leq |v|$ holds in L.

For seminormed Riesz spaces L_ρ the following properties were introduced:

(A, 0): $u_n \downarrow \theta$ in L and $\{u_n\}$ is a ρ-Cauchy sequence implies $\rho(u_n) \to 0$.

(A, i): $u_n \downarrow \theta$ in L implies $\rho(u_n) \to 0$.

(A, ii): $u_\alpha \downarrow \theta$ in L implies $\rho(u_\alpha) \to 0$.

Following Luxemburg and Zaanen [4, Notes II and XIII] we also have:

Definition 1.1 (σ-Fatou property). *A seminormed Riesz space L_ρ satisfies the σ-Fatou property whenever $\theta \leq u_n \uparrow u$ in L implies $\rho(u_n) \uparrow \rho(u)$.*

(Fatou property). *A seminormed Riesz space L_ρ satisfies the Fatou property whenever $\theta \leq u_\alpha \uparrow u$ in L implies $\rho(u_\alpha) \uparrow \rho(u)$.*

Obviously the Fatou implies the σ-Fatou, (A, i) implies the σ-Fatou and (A, ii) implies the Fatou property. Also the σ-Fatou implies the (A, 0) property. Indeed, if $\{u_n\}$ is a ρ-Cauchy sequence with $u_n \downarrow \theta$ in L, then

Received by the editors September 14, 1973.

AMS (MOS) subject classifications (1970). Primary 46A40; Secondary 46B99.

Key words and phrases. Riesz spaces, σ-Fatou property, norm completion, quotient Riesz spaces.

$\theta \leq u_m - u_n \uparrow_{n\geq m} u_m$ in L, for each fixed m, and hence $\rho(u_m - u_n) \uparrow_{n\geq m}$ $\rho(u_m)$. This implies $\rho(u_n) \to 0$.

Example 1.2. (i) Let L be the Riesz space of all real sequences which are eventually constant. Let $\rho(u) = |u(\infty)| + \sup\{|u_n|: n = 1, 2, \cdots\}$ for all $u \in L$. ($u(\infty) = u(n)$ for all sufficiently large n.) Note that the σ-Fatou property does not hold in L_ρ. However L_ρ does satisfy the (A, 0) property.

(ii) Let L be as in (i) and let $\rho(u) = \sup\{|u(n)|: n = 1, 2, \cdots\}$ for all u. Then L_ρ is noncomplete with the Fatou property. Note that (A, i) does not hold.

(iii) Let L be the Riesz space of all bounded real valued Lebesgue measurable functions defined on $[0, 1]$, with $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [0, 1]$. Let $\rho(u) = \int_0^1 |u(x)|\,dx + \sup\{|u(x)|: x \in [0, 1]\}$ for all $u \in L$. Note that L is ρ-complete with the σ-Fatou property but without the Fatou property.

(iv) The cartesian product of the spaces in (ii) and (iii) with the product norm gives a noncomplete normed Riesz space without the Fatou and (A, i) properties, but with the σ-Fatou property. □

We recall that a Riesz subspace L of a Riesz space M is said to be order dense in M if $\sup\{v \in L: \theta \leq v \leq u\} = u$ holds in M for all $u \in M^+$. If M is Archimedean (and hence so is L) then the universal completion of M [5, pp. 338–341] equally serves as the universal completion of L; consequently M can be considered as a Riesz subspace of the universal completion of L. Now, if L_ρ is a normed Riesz space with (A, 0) then L_ρ is order dense in its norm completion $\overline{L}_\rho$ [3, Theorem 61.5, p. 652] and so $\overline{L}_\rho$ "seats" in the universal completion of L as an order dense Riesz subspace. This observation will be used in the next theorem.

Theorem 1.3. *If the normed Riesz space L_ρ satisfies the σ-Fatou property, then we have:*

(i) *The norm completion $\overline{L}_\rho$ of L_ρ satisfies the σ-Fatou property.*

(ii) $\rho(u) = \inf\{\lim \rho(u_n): \{u_n\} \subseteq L^+, u_n \uparrow$ *and* $u_n \wedge |u| \uparrow |u|$ *in* $\overline{L}_\rho\}$, *for every u in $\overline{L}_\rho$.*

Proof. Let K be the universal completion of L [5, Theorem 50.8, p. 340]. Define λ on K by the formula:

$$\lambda(u) = \inf\{\lim \rho(u_n): \{u_n\} \subseteq L^+;\ u_n \uparrow \text{ and } u_n \wedge |u| \uparrow |u| \text{ in } K\}$$

with $\inf \emptyset = +\infty$. Then we have:

(i) $\lambda(u) = \rho(u)$ for all u in L.

To verify (i) use the σ-Fatou property of ρ.

(ii) $\lambda(u) = \lambda(|u|)$ for all u in K, and $\theta \leq u \leq v$ in K implies $\lambda(u) \leq \lambda(v)$.

(iii) $\lambda(u) \geq 0$ for all u in K and $\lambda(u) = 0$ implies $u = \theta$.

To see (iii) use the order density of L in K.

(iv) $\lambda(u + v) \leq \lambda(u) + \lambda(v)$, $\lambda(\alpha u) = |\alpha|\lambda(u)$ for all u, v in K and all α in $\mathbf{R}$.

(v) If $\{u_n\} \subseteq L^+$ and $\theta \leq u_n \uparrow u$ in K, then $\rho(u_n) \uparrow \lambda(u)$.

(vi) Let $U = \{u \in K^+: \theta \leq u_n \uparrow u$, for some sequence $\{u_n\} \subseteq L^+\}$. Assume $\theta \leq u_n \uparrow$ in K, $\{u_n\} \subseteq U$ and $\lambda(u_n) \uparrow \alpha < +\infty$. Then $\theta \leq u_n \uparrow u$ in K and $\lambda(u) = \alpha$ for some u in U.

To see (vi) pick $\{u_{n,k}: k = 1, 2, \cdots\} \subseteq L^+$ such that $u_{n,k} \uparrow_k u_n$ $(n = 1, 2, \cdots)$. Define $w_n = \sup\{u_{i,n}: i = 1, \cdots, n\} \in L^+$ $(n = 1, 2, \cdots)$ and note that $\rho(w_n) \leq \alpha$ for all n. Now, let $\theta < v \in L$. Pick $m \in N$ such that $m\rho(v) = \rho(mv) > \alpha$, and observe that $w_n \wedge mv \uparrow mv$ implies $\rho(mv) \leq \alpha$. So, $\sup\{w_n \wedge mv: n = 1, 2, \cdots\} < mv$. This observation implies $\theta \leq w_n \uparrow u$ in K [2, Proposition 1, p. 342]. (Since E is order dense in $C_\infty(X)$, observe that Fremlin's proof works if we replace the assumption "for every $x > 0$ in $C_\infty(X)$" by "for every $x > 0$ in E".) Thus $\theta \leq w_n \uparrow u$ and $u \in U$. Now, combine (v) with the relation $w_n \leq u_n$ for all n to obtain $\theta \leq u_n \uparrow u$ and $\lambda(u) = \alpha$.

(vii) Let $\theta \leq u$, $\lambda(u) < +\infty$ and let $\epsilon > 0$. Then there exists $v \in U$, $u \leq v$ such that $\lambda(v) \leq \lambda(u) + \epsilon$.

To verify (vii), pick $\{u_n\} \subseteq L^+$, $u_n \uparrow$, $u_n \wedge |u| \uparrow |u|$ and such that $\lim \rho(u_n) \leq \lambda(u) + \epsilon$. As in case (vi) note that $u_n \uparrow v$ in K for some v of U. Now use (v) to obtain $\lambda(v) \leq \lambda(u) + \epsilon$.

(viii) Let $L_\lambda = \{u \in K: \lambda(u) < +\infty\}$. Then L_λ is a complete normed Riesz space.

For (viii) use (vii) and a routine argument to show that L_λ satisfies the Riesz-Fischer property and hence it is λ-complete [4, Theorem 26.3, Note VIII, p. 105].

(ix) The closure of L_ρ in L_λ, $\overline{L}_\rho$, is the norm completion of L_ρ.

Now, let $\theta \leq u_n \uparrow u$ in $\overline{L}_\rho$. Since L is order dense in K, $u_n \uparrow u$ also holds in K. Given $\epsilon > 0$, pick an element u_0 in $\overline{L}_\rho$, $u \leq u_0$, $u_0 \in U$ with $\lambda(u_0 - u) < \epsilon$ (see [3, Theorem 60.3, p. 648]). Similarly pick v_n in $\overline{L}_\rho$, $u_n \leq v_n \leq u_0$, $\lambda(v_n - u_n) \leq \epsilon/2^{n+1}$ and $v_n \in U$, $n = 1, 2, \cdots$. Put $w_n = \sup\{v_i: i = 1, \cdots, n\}$ $(n = 1, 2, \cdots)$ and note $\lambda(w_n - u_n) \leq \epsilon$ and $u_n \leq w_n \leq u_0$ for all n. Hence $w_n \uparrow u_1 \leq u_0$ in L_λ and so $u \leq u_1 \leq u_0$ in L_λ. But then $\lambda(u) \leq \lambda(u_1) = \lim \lambda(w_n) \leq \lim \lambda(u_n) + \epsilon$ for all $\epsilon > 0$. Hence $\lambda(u_n) \uparrow \lambda(u)$, i.e.,

$\overline{L}_\rho$ satisfies the σ-Fatou property. Part (ii) follows immediately from the above construction. □

Corollary 1.4. *Let L_ρ be a normed Riesz space with norm completion $\overline{L}_\rho$. Then the following statements are equivalent.*

(i) *L_ρ satisfies the σ-Fatou property.*

(ii) *$\overline{L}_\rho$ satisfies the σ-Fatou property and L_ρ has* (A, 0).

Proof. To see that (ii) implies (i) use Theorem 61.5 of [3, p. 652]. □

For $L = C_{[0,1]}$ and $\rho(u) = \int_0^1 |u(x)|\,dx$ we have $\overline{L}_\rho = L_1([0, 1])$. Note that $\overline{L}_\rho$ satisfies the σ-Fatou property (in fact the (A, ii) property). However, L_ρ does not satisfy the (A, 0) property [5, Exercise 18.14(i), p. 104].

We close this section recalling a notion useful for the next section. A Riesz space L is called almost σ-Dedekind complete if it can be embedded as a super order dense Riesz subspace of a σ-Dedekind complete Riesz space K, i.e., if L is a Riesz subspace of K (more precisely L is Riesz isomorphic to a Riesz subspace of K) such that for every $\theta \le u \in K$, there exists a sequence $\{u_n\} \subseteq L$ with $\theta \le u_n \uparrow u$ in K (see [1]).

2. **The quotient Riesz space.** L_ρ/I_ρ. The null ideal of a given seminormed Riesz space L_ρ is denoted by I_ρ, i.e., $I_\rho = \{u \in L\colon \rho(u) = 0\}$. It is evident that I_ρ is a σ-ideal (resp. a band) if ρ satisfies the σ-Fatou property (resp. the Fatou property). It is also obvious that the quotient Riesz space L_ρ/I_ρ becomes a normed Riesz space under the norm $[\rho]([u]) = \rho(u)$. ($[u]$ denotes the equivalence class of u.)

Question: If L_ρ satisfies the σ-Fatou property, does the normed Riesz space L_ρ/I_ρ satisfy the σ-Fatou property?

The next theorem gives a condition for the answer to be affirmative.

Theorem 2.1. *Assume that the seminormed Riesz space L_ρ satisfies the σ-Fatou property and that L is almost σ-Dedekind complete. Then the normed Riesz space L_ρ/I_ρ satisfies the σ-Fatou property.*

Proof. Let K be a σ-Dedekind complete Riesz space containing L as a super order dense Riesz subspace. We can assume that the ideal generated by L is all of K. Given $u \in K$ pick $\{u_n\} \subseteq L$ with $\theta \le u_n \uparrow |u|$ in K and define $\lambda(u) = \lim \rho(u_n)$. Note that $\lambda(u)$ is independent of the sequence chosen and that λ is a Riesz seminorm of K with the σ-Fatou property and with $\lambda = \rho$ on L.

Let L/I_λ be the canonical image of L in K_λ/I_λ. Observe that L_ρ/I_ρ

is Riesz isomorphic to L/I_λ (the mapping $[u] = u + I_\rho \to u + I_\lambda = [u]$ does it) and that the quotient norm $[\rho]$ on L_ρ/I_ρ and the norm induced from K_λ/I_λ to L/I_λ coincide. Now let $[\theta] \le [u_n] \uparrow [u]$ in L_ρ/I_ρ, so $[\theta] \le [u_n] \uparrow [u]$ holds also in L/I_λ. We can assume $\theta \le u_n \uparrow \le u$ in L, so $\theta \le u_n \uparrow v \le u$ holds in K and hence $[\theta] \le [u_n] \uparrow [v]$ in K_λ/I_λ [5, Theorem 18.11, p. 103]. Since L/I_λ is order dense in K_λ/I_λ, $[u_n] \uparrow [u]$ also holds in K_λ/I_λ and hence $[v] = [u]$, so $\lambda(v) = \lambda(u) = \rho(u)$.

Thus $[\rho]([u_n]) = \rho(u_n) = \lambda(u_n) \uparrow \lambda(v) = \rho(u) = [\rho]([u])$, and the proof is finished. □

Question: If we replace the almost σ-Dedekind completeness of L by Archimedeanness is Theorem 2.1 still true?

The following example shows that the answer is negative in general.

Example 2.2. Let L be the Archimedean Riesz space $C(\mathbf{R}_\infty)$. ($\mathbf{R}_\infty$ is the one point compactification of the real numbers considered with the discrete topology (see [5, Example (v), p. 141]). Note that L is not almost σ-Dedekind complete. Now, define the Riesz seminorm ρ on L, by $\rho(u) = |u(\infty)| + \sup\{|u(n)|: n = 1, 2, \cdots\}$. Note that ρ satisfies the σ-Fatou property but not the Fatou property. (In fact ρ satisfies the (A, i) property.) Note also that I_ρ is a band.

Now, let $u_n = \chi_{\{1,\cdots,n\}}$, $n = 1, 2, \cdots$. Then $\theta \le u_n \uparrow \le e$ in L ($e(x) = 1$ for all $x \in \mathbf{R}$) and $\rho(u_n) = 1$ for all n. It is easily seen that $[\theta] \le [u_n] \uparrow [e]$ holds in L_ρ/I_ρ. But

$$[\rho]([u_n]) = \rho(u_n) = 1 \not\uparrow [\rho]([e]) = \rho(e) = 2.$$

Hence L_ρ/I_ρ does not satisfy the σ-Fatou property. □

A better situation holds if ρ satisfies the Fatou property. The next theorem tells us that L_ρ/I_ρ satisfies the Fatou property if L_ρ does.

Theorem 2.3. *Let L_ρ be an Archimedean seminormed Riesz space with the Fatou property. Then the normed Riesz space L_ρ/I_ρ satisfies the Fatou property.*

Proof. Repeat the proof of Theorem 2.1 replacing K by L^δ, the Dedekind completion of L. □

REFERENCES

1. C. D. Aliprantis and Eric Langford, *Almost σ-Dedekind complete Riesz spaces and the main inclusion theorem*, Proc. Amer. Math. Soc. 44 (1974), 421–426.

2. D. H. Fremlin, *On the completion of locally solid vector lattices*, Pacific J. Math. 43 (1972), 341–347.

3. W. A. J. Luxemburg, *Notes on Banach function spaces*. XVIa, Nederl. Akad. Wetensch. Proc. Ser. A 68 = Indag. Math. 27 (1965), 646–657. MR 32 #6202e.

4. W. A. J. Luxemburg and A. C. Zaanen, *Notes on Banach function spaces*. II, VIII, XIII, Nederl. Akad. Wetensch. Proc. Ser. A 66 = Indag. Math. 25 (1963), 148–153; ibid. A 67 = Indag. Math. 26 (1964), 104–119, 530–543. MR 26 #6723b; 28 #5324b; 30 #3381b.

5. ———, *Riesz spaces*. I, North-Holland, Amsterdam, 1971.

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91109

Current address: STD Research Corporation, 150 E. Foothill, Arcadia, California 91006 and Department of Mathematics, Occidental College, Los Angeles, California 90041