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BOUNDED SOLUTIONS OF THE EQUATION $\Delta u = pu$ ON A RIEMANNIAN MANIFOLD

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ABSTRACT. Given a nonnegative C^1-function $p(x)$ on a Riemannian manifold R, denote by $B_p(R)$ the Banach space of all bounded C^2-solutions of $\Delta u = pu$ with the sup-norm. The purpose of this paper is to give a unified treatment of $B_p(R)$ on the Wiener compactification for all densities $p(x)$. This approach not only generalizes classical results in the harmonic case ($p \equiv 0$), but it also enables one, for example, to easily compare the Banach space structure of the spaces $B_p(R)$ for various densities $p(x)$. Typically, let $\beta(p)$ be the set of all p-potential nondensity points in the Wiener harmonic boundary Δ, and $C_p(\Delta)$ the space of bounded continuous functions f on Δ with $f|\Delta - \beta(p) \equiv 0$.

Theorem. *The spaces $B_p(R)$ and $C_p(\Delta)$ are isometrically isomorphic with respect to the sup-norm.*

Throughout this paper R is an orientable Riemannian C^∞-manifold of dim ≥ 2, and $p(x)$ is a nonnegative C^1-function on R. Denote by $B_p(R)$ the space of bounded C^2-solutions u on R of the elliptic equation $\Delta u = pu$, where Δu is the Laplacian of u on R. As one studies bounded harmonic functions on the Wiener compactification, the space $B_p(R)$ has been investigated on the so-called Wiener p-compactification (cf. Loeb and Walsh [2], Wang [9]). However, their consideration restricts one to construct different compactifications for different densities $p(x)$.

The purpose of the present paper is to give a unified treatment of the spaces $B_p(R)$ on the Wiener compactification R^* for all densities $p(x)$. This approach, for instance, enables one to easily compare the linear space structure of the spaces $B_p(R)$ for various densities $p(x)$. Typically, let $\beta(p)$ be the set of p-potential nondensity points x in the Wiener harmonic boundary Δ (see below for its definition), and $C_p(\Delta)$ the space of bounded

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continuous functions f on Δ such that $f|\Delta - \beta(p) \equiv 0$. Then $B_p(R)$ and $C_p(\Delta)$ are isometrically isomorphic with respect to the sup-norm.

For the notation and terminology we refer the reader to Sario and Nakai [8, Chapter 4].

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1. First we observe a simple fact.

Lemma. *Every $u \in B_p(R)$ is continuously extendable to the Wiener compactification R^* of R. Furthermore u has the property $\|u\| = \max_\Delta |u|$, where $\|\cdot\|$ is the sup-norm and Δ is the Wiener harmonic boundary.*

A point $x \in \Delta$ will be called a p-potential nondensity point if there exists an open neighborhood U^* of x in R^* such that

$$\sup_{a \in U} \int_U G_U(a, y) p(y)\, dy < \infty,$$

where $U = U^* \cap R$, $G_U(a, y)$ is the (harmonic) Green's function for U, and dy is the (Riemannian) volume element of R. Denote by $\beta(p)$ the set of all p-potential nondensity points in Δ (cf. Nakai [5]).

For $p \not\equiv 0$ the above maximum principle is too crude for our purpose:

Theorem. *Every $u \in B_p(R)$ has the property $\|u\| = \max_{\beta(p)} |u|$.*

Proof. It suffices to show that $u \equiv 0$ on $\Delta - \beta(p)$. To the contrary suppose that $u(x) = 2\epsilon > 0$ for some $x \in \Delta - \beta(p)$. Choose an open neighborhood U^* of x in R^* such that $u > \epsilon$ on U^*. Set $U = U^* \cap R$. We may modify U to have a smooth ∂U. Let $\{\Omega_n\}_1^\infty$ be a "regular" exhaustion of U. By Stokes' formula

$$u(z) = h_n(z) - \int_{\Omega_n} G_n(z, y) p(y) u(y)\, dy$$

on Ω_n, where $h_n \in B_0(\Omega_n)$ with $h_n|\partial\Omega_n \equiv u$ and $G_n(z, y)$ is the Green's function for Ω_n. Therefore it is seen that

$$0 \le \int_{\Omega_n} G_n(z, y) p(y) u(y)\, dy \le \|u\|$$

on Ω_n. By the monotone convergence theorem, we deduce that

$$\int_U G_U(z, y) p(y)\, dy \le \|u\|/\epsilon$$

on U, a contradiction to the fact that $x \notin \beta(p)$.

2. For a parabolic R, the space $B_p(R) = \{0\}$ or the real number field according as $p \not\equiv 0$ or $p \equiv 0$ (Ozawa [6]). We thus assume that R is hyperbolic. Set

$$H_pB(R) = \{u \in B_0(R) | u \equiv 0 \text{ on } \Delta - \beta(p)\}.$$

It is not difficult to see that $C_p(\Delta)$ and $H_pB(R)$ are isometrically isomorphic Banach spaces with the sup-norm $\|\cdot\|$.

Theorem. *For any density $p(x)$ on R the Banach spaces $B_p(R)$ and $C_p(\Delta)$ are isometrically isomorphic. In particular*

$$C_p(\Delta) = \{u|\Delta\colon u \in B_p(R)\}.$$

Proof. It suffices to show that every $h \in C_p(\Delta)$ can be extended to a function in $B_p(R)$.

Without loss of generality we may assume that $h \in H_pB(R)$ and $h \geq 0$ on R. Define $v(z) = \sup \{u(z) | u \in F_h\}$, where $F_h = \{u \in B_p(R) | 0 \leq u \leq h$ on $R\}$. Since the class F_h forms a Perron family for $\Delta u = pu$, it follows that $v \in B_p(R)$. We need to prove that $v \equiv h$ on $\beta(p)$.

On the contrary, assume that there exists a point $x \in \beta(p)$ such that $h(x) > v(x) \geq 0$. Let ϵ be a positive constant with $v(x) < \epsilon < h(x)$. Choose an open neighborhood U^* of x in R^* such that $h > \epsilon > v$ on U^*, $U = U^* \cap R$ has smooth ∂U, and $\sup_{a \in U} \int_U G_U(a, y)p(y)\, dy < \infty$. Take n so large that $\sup_{a \in U} \int_U G_U(a, y)p(y)\, dy < n$.

For any $\phi \in C(U)$, the space of bounded continuous functions on U, define an integral operator T by

$$(T\phi)(z) = -\frac{1}{n} \int_U G_U(z, y)p(y)\phi(y)\, dy.$$

It is well known (cf. e.g. Miranda [3, p. 25]) that T is a linear operator in $C(U)$ and its operator norm satisfies

$$\|T\| \leq \frac{1}{n} \sup_{a \in U} \int_U G_U(a, y)p(y)\, dy < 1.$$

Thus the Fredholm integral equation $(I - T)u = k$ has a unique solution u, where I is the identity operator in $C(U)$ and $k \in B_0(U)$ such that $k|\partial U \equiv 0$, $0 \leq k \leq h$ on U, and $k(x) = h(x)$. Clearly $u \in B_{n^{-1}p}(U)$, $u \equiv k$ on $(\partial U) \cup (U^* \cap \Delta)$, and $0 \leq u \leq k$ on U. Extend u to R by setting $u|R - U \equiv 0$ and then construct $u_0 \in B_{n^{-1}p}(R)$ such that $u \leq u_0 \leq \pi u$ on R. Here πu is the harmonic projection of u on R. Note that $u_0(x) = h(x)$, $u_0 \equiv 0$ on $\Delta - U^*$, and $u_0 \leq h$ on R.

Let $\{R_i\}_1^\infty$ be a regular exhaustion of R and take $w_i \in B_p(R_i)$ such that $w_i \equiv u_0$ on $R - R_i$. In view of $\Delta u_0 = n^{-1}pu_0 \le pu_0$ on R, it is not difficult to see that $0 \le w_i \le w_{i+1} \le u_0$ on R and $\|w_i\| = \|u_0\|$. By Harnack's principle for $\Delta u = pu$, the sequence $\{w_i\}$ converges, uniformly on a compact subset of R, to a function $w \in B_p(R)$, such that $0 \le w \le u_0$ on R and $\|w\| = \|u_0\|$. Since $w \equiv 0$ on $\Delta - U^*$, we conclude that

$$\max_{U^*\cap\Delta} w = \|w\| = \|u_0\| = \max_{U^*\cap\Delta} u_0 \ge u_0(x) = h(x) > \epsilon.$$

In view of $w \in F_h$, $\max_{U^*\cap\Delta} v \ge \max_{U^*\cap\Delta} w > \epsilon$. But this contradicts our choice of U^*: $v < \epsilon$ on U^*.

This completes the proof of our theorem.

3. As an application of our theorem we would like to mention its contribution to the comparison problem of the spaces $B_p(R)$ for various densities $p(x)$. In this vein an elegant result of Royden [7] states: if $p(x)$ and $q(x)$ are two densities such that for some constant $\alpha \ge 1$, $\alpha^{-1}p(x) \le q(x) \le \alpha p(x)$ off some compact subset of R, then $B_p(R)$ and $B_q(R)$ are isometric. Later Nakai [4] found another important criterion: the same conclusion holds if $\int_R |p(x) - q(x)|\, dx < \infty$.

The following result considerably sharpens their conclusions in view of the fact that Δ is topologically "small" in $R^* - R$.

Corollary. *Banach spaces $B_p(R)$ and $B_q(R)$ are isometrically isomorphic in each of the following cases*:

(i) *there exists a constant $\alpha \ge 1$ such that $\alpha^{-1}p(x) \le q(x) \le \alpha p(x)$ in some open neighborhood U^* of Δ in R^**;

(ii) *there exists an open neighborhood U^* of Δ in R^* such that $\int_{U^*\cap R} |p(x) - q(x)|\, dx < \infty$.*

Proof. In case (i), it is easy to see that $\beta(p) = \beta(q)$. Now assume that condition (ii) holds. Contrary to our conclusion, suppose that there exists a point $a \in \beta(p) - \beta(q)$. In this case there exists a $u \in B_p(R)$ such that $0 < u < 1$ on R and $u(a) = 1$. Since $a \notin \beta(q)$, $v(a) = 0$ for $v \in B_q(R)$. Choose an open neighborhood V^* of a in R^* such that

$$\sup_{x\in V} \int_V G_V(x, y)p(y)\, dy < \infty \quad \text{and} \quad \int_V |p(x) - q(x)|\, dx < \infty,$$

where $V = V^* \cap R$. It can be shown from the second inequality that

$$\int_V G_V(x, y)|p(y) - q(y)|\, dy < \infty$$

for each $x \in V$. For a "regular" exhaustion $\{\Omega_n\}_1^\infty$ of V construct v_n on V such that $v_n \in B_q(\Omega_n)$ and $v_n \equiv u$ on $V - \Omega_n$. Then we can write

$$u(x) = h_n(x) - \int_{\Omega_n} G_n(x, y)p(y)u(y)\,dy,$$

$$v_n(x) = h_n(x) - \int_{\Omega_n} G_n(x, y)q(y)v_n(y)\,dy$$

on Ω_n, where $h_n \in B_0(\Omega_n)$ with $h_n|\partial\Omega_n \equiv u$ and $G_n(x, y)$ is the Green's function on Ω_n. Since $0 \le h_n \le 1$ and $0 \le v_n \le 1$, we may assume that $h_n \to h \in B_0(V)$ and $v_n \to v \in B_q(V)$, uniformly on compact subsets of V. In view of

$$\begin{aligned}|u(x) - v_n(x)| &\le \int_{\Omega_n} G_n(x, y)|q(y) - p(y)|v_n(y)\,dy \\ &\quad + \int_{\Omega_n} G_n(x, y)p(y)|v_n(y) - u(y)|\,dy \\ &\le \int_{\Omega_n} G_V(x, y)[|q(y) - p(y)| + p(y)]\,dy\end{aligned}$$

we conclude that

$$|u(x) - v(x)| \le \int_V G_V(x, y)[|q(y) - p(y)| + p(y)]\,dy$$

on V and therefore on $V \cup \{a\}$. Note that all three functions in the above inequality have continuous extensions to V^*. Since the Green's potential vanishes on $V^* \cap \Delta$, we deduce that $v(a) = u(a) = 1$, a contradiction to the fact that $a \notin \beta(q)$.

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