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AN ALMOST CONTINUOUS FUNCTION

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ABSTRACT. In this note, a function is constructed which is of the Cesàro type and is almost continuous in the sense of Stallings and in the sense of Husain.

Introduction. Bruce D. Smith [1, p. 318] has shown that a real function $f\colon R \to R$ is continuous if, and only if: (i) it is almost continuous in the sense of Stallings, (ii) it is almost continuous in the sense of Husain, and (iii) it is not of the Cesàro type. He gives examples to show that conditions (i) and (ii) are not redundant. An example is given that shows (iii) is independent, provided that functions which have connected graphs are almost continuous in the sense of Stallings. We first note that examples of functions with connected graphs which are not almost continuous in the sense of Stallings are given in [2], [3], [4], [5]. The example given here establishes the independence of condition (iii).

Throughout the following, R will denote the real numbers with the standard topology, $G(f) \subset R \times R$ will denote the graph of $f\colon R \to R$. The projection of $A \subset R \times R$ on the X-axis will be denoted by $(A)_X$ and the boundary of A by $\mathrm{Bd}(A)$. For a set A, $\mathrm{Cl}(A)$ denotes the closure of A.

In the following three definitions, X and Y are topological spaces, f a function from X to Y.

Definition 1. The function f is *almost continuous in the sense of Stallings* if, and only if, for any open set $N \subset X \times Y$, if $G(f) \subset N$, then N contains the graph of a continuous function.

Definition 2. The function f is *almost continuous in the sense of Husain* if, and only if, for each $x \in X$, if $V \subset Y$ is an open set containing $f(x)$, then $\mathrm{Cl}(f^{-1}(V))$ is a neighborhood of x.

Definition 3. The function f is *of the Cesàro type* if, and only if, there

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exist nonempty open sets, $U \subset X$ and $V \subset Y$, such that, for each $y \in V$, $U \subset \mathrm{Cl}(f^{-1}(y))$.

It is clear from the definitions that any function $f\colon R \to R$ which assumes every real value in every interval is almost continuous in the sense of Husain and is of the Cesàro type.

Lemma. *If $N \subset R \times R$ is an open set which contains the graph of a function $h\colon R \to R$ and $(\mathrm{Bd}(N))_X$ is of the first category, then N contains the graph of a continuous function.*

Proof. Let I be a compact, nondegenerate interval on the X-axis. Since $G(h) \subset N$ and N is open, for each x_0 in I, we may choose a horizontal line segment $L_{x_0} = \{(x, y) | a_0 < x < b_0;\ y = h(x_0)\}$ which contains the point $(x_0, h(x_0))$ and lies entirely in N. Then $\{(L_{x_0})_X\colon x_0 \in I\}$ is an open cover of I and can be reduced to $(L_{x_1})_X, (L_{x_2})_X, \cdots, (L_{x_n})_X$ where $x_1 < x_2 < \cdots < x_n$ and no three intervals overlap.

There exists a compact interval J on the Y-axis such that $L_{x_i} \subset I \times J$ for $i = 1, 2, \cdots, n$. Since $(\mathrm{Bd}(N))_X$ is of the first category, $(\mathrm{Bd}(N) \cap (I \times J))_X$ is compact and nowhere dense. Hence, $(L_{x_i})_X \cap (L_{x_{i+1}})_X$ contains an interval K_i which contains no points of $(\mathrm{Bd}(N) \cap (I \times J))_X$.

Assume $h(x_i) < h(x_{i+1})$. The rectangle lying above K_i and between the segments L_{x_i} and $L_{x_{i+1}}$ contains no points of $\mathrm{Bd}(N)$ and, therefore, no points of $R \times R - N$. Hence, we may join L_{x_i} and $L_{x_{i+1}}$ with a line segment $y = M_i(x)$ with positive slope which lies in N. If $h(x_i) > h(x_{i+1})$, we join L_{x_i} and $L_{x_{i+1}}$ with a line segment $y = M_i(x)$ with negative slope which lies in N. If $h(x_i) = h(x_{i+1})$, no connection is necessary and we take $M_i(x) = L_{x_i} \cap L_{x_{i+1}}$.

Let

$$f(x) = \begin{cases} h(x_i) & \text{if } x \in (L_{x_i})_X - (K_{i-1} \cup K_i), \\ M_i(x) & \text{if } x \in K_i. \end{cases}$$

Then $f(x)$ is continuous on I and $G(f) \subset N$. Let $I_n = [n, n+2]$ for each integer n. By joining the associated continuous functions in the manner described above, we extend f to the entire real line and the Lemma is proved. We proceed to the construction of the promised example.

Definition. An open set $M \subset R \times R$ will be called *special* if there exist numbers a_0, b_0, and y_0, with $a_0 < b_0$, such that

$$R \times R - M = \{(a_0, y) | y < y_0\} \cup \{(x, y_0) | a_0 \leq x \leq b_0\}$$
$$\cup \{(b_0, y) | y > y_0\}.$$

Let ω_c denote the first ordinal of cardinality c, where c denotes the power of the continuum. Let $M_1, M_2, \cdots, M_\alpha, \cdots, \alpha < \omega_c$ be a well-ordering of the open sets in the plane which contain the graph of a function, but do not contain the graph of any continuous function.

It is easily seen that every special set appears in this list. By the Lemma, for each $\alpha < \omega_c$, $(\mathrm{Bd}(M_\alpha))_X$ is of the second category and, being a Borel set, must contain c points.

By transfinite induction we may choose for each $\alpha < \omega_c$ a point $(x_\alpha, f(x_\alpha))$ in $\mathrm{Bd}(M_\alpha)$ in such a way that

(i) $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$, and

(ii) if M_α is special, $(x_\alpha, f(x_\alpha))$ is on the interior of the horizontal segment of $\mathrm{Bd}(M_\alpha)$.

If $x \in R$ and $x \neq x_\alpha$ for any $\alpha < \omega_c$, set $f(x) = 0$.

By (ii) $f(x)$ assumes every real value on every interval and is, therefore, almost continuous in the sense of Husain and of the Cesàro type.

If $M \subset R \times R$ is an open set containing $G(f)$, then $M \neq M_\alpha$ for any $\alpha < \omega_c$ since $(x_\alpha, f(x_\alpha))$ was chosen in $\mathrm{Bd}(M_\alpha)$. Therefore, M contains the graph of a continuous function and $f(x)$ is continuous in the sense of Stallings.

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