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PROXIMITY SPACES AND TOPOLOGICAL FUNCTORS

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ABSTRACT. The purpose of this paper is to determine what natural functors $T: \mathbf{A} \to \mathbf{X}$ are $(\mathfrak{E}, \mathfrak{M})$-topological, where $\mathbf{A}$ is a subcategory of the category of proximity or uniform spaces and $\mathbf{X}$ is an $(\mathfrak{E}, \mathfrak{M})$-category. We give necessary and sufficient conditions under which a point separating family of continuous functions can be nicely lifted to a proximally continuous family. Proximities having a finest compatible uniform structure are characterized.

Introduction. A problem which frequently arises in analysis is to determine whether or not a given continuous function is proximally (uniformly) continuous. More generally, let X be a topological space and let F be a family of continuous functions, each member f of F being from X into a proximity space Y_f. Does there exist a "compatible" proximity structure δ on X satisfying:

(i) each $f \in F$ is p-continuous on (X, δ), and

(ii) for any proximity space Z and any continuous function g from Z to X, g is p-continuous iff fg is p-continuous for each $f \in F$.

Problems similar to this are solved in standard textbooks on topology with X a set, F a family of functions and Y_f either a uniform space or a topological space; however, more general problems of this type have not yet been discussed in the literature.

In order to study these questions it is convenient to make use of the notions of $(\mathfrak{E}, \mathfrak{M})$-categories and topological functors as introduced by Herrlich [2].

1. **Preliminaries.** Let $\mathbf{X}$ be a category. A *source* in $\mathbf{X}$ is a pair $(X, f_i)_I$, where X is an object in $\mathbf{X}$ and $f_i: X \to X_i$ is a family of $\mathbf{X}$-morphisms indexed by a class I. $(X, f_i)_I$ is a *monosource* if it is a source, and $r = s$ whenever $f_i r = f_i s$ for every $i \in I$. A morphism e in a category $\mathbf{X}$ is a *regular epimor-*

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phism if and only if it is the coequalizer of some pair of morphisms in $\mathbf{X}$. A category $\mathbf{X}$ is an $(\mathfrak{E}, \mathfrak{M})$-category provided that $\mathfrak{E}$ is a class of epimorphisms in $\mathbf{X}$, closed under composition with isomorphisms, $\mathfrak{M}$ is a class of sources in $\mathbf{X}$, closed under composition with isomorphisms, and the following conditions hold:

(a) For every source $(X, f_i)_I$ in $\mathbf{X}$ there exists e in $\mathfrak{E}$ and $(Y, m_i)_I$ in $\mathfrak{M}$ such that $f_i = m_i e$ for each $i \in I$, and

(b) whenever f and e are morphisms and $(Y, m_i)_I$ and $(Z, f_i)_I$ are sources in $\mathbf{X}$ such that $e \in \mathfrak{E}$, $(Y, m_i) \in \mathfrak{M}$ and $f_i e = m_i f$ for each $i \in I$, then there exists a (unique) morphism g in $\mathbf{X}$ such that $f = ge$ and $f_i = m_i g$ for each $i \in I$.

(1.1) **Definition.** Let $\mathbf{X}$ be an $(\mathfrak{E}, \mathfrak{M})$-category and let $T: \mathbf{A} \to \mathbf{X}$ be a functor.

(i) A source $(A, f_i)_I$ in $\mathbf{A}$ is called *T-initial* provided for any source $(B, g_i)_I$ in $\mathbf{A}$ and any $\mathbf{X}$-morphism $f: TB \to TA$ with $Tf_i \cdot f = Tg_i$, for each $i \in I$, there exists a unique $\mathbf{A}$-morphism $h: B \to A$ in $\mathbf{A}$ such that $Th = f$ and $f_i h = g_i$ for each $i \in I$.

(ii) A source $(A, f_i: A \to A_i)_I$ *T-lifts* a source $(X, g_i: X \to TA_i)_I$ in $\mathbf{X}$ if and only if there exists an isomorphism $k: X \to TA$ in $\mathbf{X}$ with $Tf_i k = g_i$ for each $i \in I$.

(iii) A source $(A, f_i)_I$ in $\mathbf{A}$ is *initial* to a source $(X, Tf_i)_I$ in $\mathbf{X}$ if and only if $(A, f_i)_I$ is T-initial and T-lifts $(X, Tf_i)_I$. When the meaning is clear from the context, we say that A is initial to the source $(X, Tf_i)_I$.

(iv) T is *$(\mathfrak{E}, \mathfrak{M})$-topological* if and only if for each family $(A_i)_I$ of $\mathbf{A}$-objects and each source $(X, m_i: X \to TA_i)_I$ in $\mathfrak{M}$ there exists a T-initial source $(A, f_i: A \to A_i)_I$ in $\mathbf{A}$ which T-lifts $(X, m_i)_I$.

(v) T is called *absolutely topological* if and only if it is $(\mathfrak{E}, \mathfrak{M})$-topological for any $(\mathfrak{E}, \mathfrak{M})$-structure on $\mathbf{X}$.

For the definitions of LO- and EF-proximities, the reader is referred to [1]. The definition of a LO-base is analogous to that of an EF-base given in [5].

If δ is a LO-proximity on X then (X, δ) is called a LO-space (respectively an EF-space). For two LO-spaces (X, θ) and (Y, δ), a function f on X to Y is said to be p-continuous provided $(A, B) \in \theta$ implies $(fA, fB) \in \delta$.

For two binary relations θ_1 and θ_2 on $\mathcal{P}(X)$, we say that θ_1 is *finer than* θ_2 (θ_2 *is coarser than* θ_1) if and only if $\theta_1 \subset \theta_2$. For a given LO-space (X, δ) and a subset $A \subset X$, defining $\overline{A} = \{x \in X: (\{x\}, A) \in \delta\}$ we get a Kuratowski closure operator on X. The induced topology is always R_0. For a

R_0-space $(X, \mathcal{T})$ there are two distinguished LO-proximities. The finest LO-proximity which induces $\mathcal{T}$ is denoted by δ_0 and is given by $(A, B) \in \delta_0$ if and only if $\overline{A} \cap \overline{B} \neq \emptyset$. A subset of the form $\overline{\{x\}}$ is called a point closure. The coarsest LO-proximity inducing $\mathcal{T}$ is denoted by δ_c and is given by: $(A, B) \in \delta_c$ if and only if $\overline{A} \cap \overline{B} \neq \emptyset$ or each of the sets $\overline{A}$ and $\overline{B}$ is a union of infinitely many distinct point-closures [4].

Suppose β is a LO-base on a set X. Define $\delta(\beta)$ as follows: $(A, B) \notin \delta(\beta)$ if and only if there are finite covers $\{A_i: 1 \leqq i \leqq n\}$ and $\{B_j : 1 \leqq j \leqq m\}$ of A and B respectively such that $(A_i, B_j) \notin \beta$ for any i, j. Then $\delta(\beta)$ is the coarsest LO-proximity finer than the base β. The LO-proximity (respectively EF-proximity) $\delta(\beta)$ is said to be *generated by* the base β. Note that if β is an EF-base then $\delta(\beta)$ is an EF-proximity.

(1.3) **Lemma.** *Let β_1, β_2 be LO-bases on sets X and Y respectively, and let f be a function on X to Y such that $(A, B) \in \beta_1$ implies $(fA, fB) \in \beta_2$. Then $f: (X, \delta(\beta_1)) \to (Y, \delta(\beta_2))$ is p-continuous.*

For any collection $S = \{\delta_i: i \in I\}$ of LO-proximities on a set X, $\bigvee S$ denotes the join of the collection S. If each δ_i is EF then so is $\bigvee \delta_i$. The following result is true for LO as well as EF-proximities.

(1.4) **Lemma.** *Let $f: (X, \delta_\lambda) \to (Y, \delta'_\lambda)$ be p-continuous for each member λ of some indexing set I. Then $f: (X, \bigvee\delta_\lambda) \to (Y, \bigvee\delta'_\lambda)$ is p-continuous.*

Proof. Let $\beta_1 = \bigcap\{\delta_\lambda: \lambda \in I\}$ and $\beta_2 = \bigcap\{\delta'_\lambda: \lambda \in I\}$. Then for each $(A, B) \in \beta_1$, $(f(A), f(B)) \in \beta_2$ and the conclusion follows from Lemma 1.3.

(1.5) **Lemma.** *Let F be a family of functions, each member f of F being on a set X into a LO-space (Y_f, δ_f). Then there exists a coarsest LO-proximity δ_F on X making each member of F p-continuous. Moreover δ_F is compatible with the weak topology on X determined by the family F; and if each δ_f is EF then so is δ_F.*

Proof. Define a binary relation β_F on the power-set of X as follows: $(A, B) \in \beta_F$ if and only if $(fA, fB) \in \delta_f$ for each f in F. β_F is a LO-base on X and the LO-proximity $\delta_F = \delta(\beta_F)$ fulfills all the requirements.

The relation δ_F is called the *weak-LO* (respectively *weak-EF*) proximity on X determined by the family F. By **LO** (respectively **EF**) we denote the category of LO-spaces (respectively EF-spaces) and p-continuous maps. **S** is the category of sets.

2. Topological functors.

(2.1) **Theorem.** *The functors* (**LO** – S) *and* (**EF** – S) *are absolutely topological.*

Proof. Let $(Y_i, \delta_i)_I$ be a family of LO-spaces and $(X, f_i\colon X \to Y_i)_I$ be any source in S. Denote by δ_F the weak LO-proximity on X determined by the family $F = (f_i)_I$. Let (Z, δ) be a LO-space and suppose $g\colon (Z, \delta) \to (X, \delta_F)$ is not p-continuous. Take $(A, B) \in \delta$ such that $(gA, gB) \notin \delta_F$. Since $\delta_F = \delta(\beta_F)$ there exist finite covers $\{H_i\colon 1 \leqq i \leqq m\}$ and $\{K_j\colon 1 \leqq j \leqq n\}$ of $g(A)$ and $g(B)$ respectively such that $(H_i, K_j) \notin \beta_F$ for any i, j. Also there exist i, j such that $(g^{-1}H_i, g^{-1}K_j) \in \delta$; and as $(H_i, K_j) \notin \beta_F$ there is $f_k \in F$ such that $(f_kH_i, f_kK_j) \notin \delta_k$. This shows that $f_k g$ is not p-continuous. The proof is now complete.

(2.2) **Theorem.** *For any full epireflective subcategory* **C** *of* **LO** *the functor* (**C** – S) *is (epi, mono-source) topological.*

Proof. Let $(Y_i)_I$ be a family of objects in **C** and let $(X, m_i\colon X \to Y_i)_I$ be a mono-source in S. Denote by δ_I the weak LO-proximity on X induced by the family $\{m_i\colon i \in I\}$. The evaluation map on (X, δ_I) is a proximal isomorphism into the product of $(Y_i)_I$. Thus, (X, δ_I) belongs to **C**. Obviously, the source $((X, \delta_I), m_i)$ is initial to (X, m_i) in S and the proof is complete.

By **LO*** (respectively **EF***) we denote the full subcategory of **LO** consisting of all separated LO (respectively separated EF) spaces. Since **LO***, **EF**, and **EF*** are full epireflective subcategories of **LO**, we have the following:

(2.3) **Corollary.** *Each of the functors* (**LO*** – S), (**EF** – S) *and* (**EF*** – S) *is (epi, mono-source) topological.*

(2.4) **Example.** The functor **LO*** – S is not absolutely topological.

Proof. Let X be an infinite set. Take $X \times X = Z$ with the LO-proximity δ_1 defined by $(P, Q) \in \delta_1$ if and only if $P \cap Q \neq \emptyset$ or each of the sets P and Q is infinite. Consider the function $\pi\colon Z \to X$ defined by $\pi(x_1, x_2) = x_1$ for all $(x_1, x_2) \in Z$. Then for any two disjoint infinite subsets A and B of X we have $(b \times A, a \times B) \in \delta_1$, for $b \in B$ and $a \in A$; but $(b, a) \notin \delta$ for any separated LO-proximity δ on X. Thus no object in **LO*** can be initial to $(X, f_0\colon X \to X)$ where f_0 is a constant function on X to X.

We remark that none of the functors (**LO** – $\mathbf{R}_0$), (**LO*** – $\mathbf{T}_1$), (**EF** – **CR**) (**EF*** – $\mathbf{T}_{3\frac{1}{2}}$) is absolutely topological. This can be easily seen by considering such examples as the following.

(2.5) **Example.** For any nonindiscrete R_0-space X there exists a LO-space Z and a continuous function f on Z to X such that for no compatible LO-proximity on X is f p-continuous.

Proof. Let Y be an infinite discrete topological space and set $Z = X \times Y$. The projection p_x of Z into X is continuous. Let δ_c be the coarsest compatible LO-proximity on Z. Since X is nonindiscrete R_0, there exist two points x_1, x_2 of X such that $\mathrm{cl}(\{x_1\}) \cap \mathrm{cl}(\{x_2\}) = \varnothing$, and therefore for any compatible LO-proximity δ on X, $(\{x_1\}, \{x_2\}) \notin \delta$. Also $p_x^{-1}\mathrm{cl}(\{x_1\}) = \{(a, y): a \in \mathrm{cl}(\{x_1\}), y \in Y\}$ and thus each of the sets $p_x^{-1}\mathrm{cl}(\{x_1\})$, $p_x^{-1}\mathrm{cl}(\{x_2\})$ is the union of infinitely many pairwise disjoint point-closures, thereby showing that $(p_x^{-1}\mathrm{cl}(\{x_1\}), p_x^{-1}\mathrm{cl}(\{x_2\})) \in \delta_c$. Thus the function p_x cannot be p-continuous.

Let X be a topological space. A *determinator on* X is a source (X, f_i) in **TOP** such that for each closed set A in X and each point x in the complement of A we can find a finite subfamily $\{f_i: 1 \leqq i \leqq n\}$ of F and a finite cover $\{A_i: 1 \leqq i \leqq n\}$ of A such that $f_i(x) \notin \mathrm{cl}(f_i(A_i))$ for any i. Observe that any family of continuous functions which distinguishes points and closed sets is a determinator on X.

(2.6) **Theorem.** *The weak topology determined on a topological space X by a source of continuous functions on X coincides with the given topology of X if and only if the source is a determinator on X.*

(2.7) **Theorem.** *Let (Y_i, δ_i) be a collection of separated LO-spaces and $(X, f_i: X \to Y_i)$ a mono-source in* $\mathbf{T}_1$. *A source initial to $(X, f_i)_I$ via the functor* $(\mathbf{LO}^* - \mathbf{T}_1)$ *exists in* $\mathbf{LO}^*$ *if and only if $F = (X, f_i)_I$ is a determinator on X.*

Proof. If F is a determinator on X then by Lemma 1.5 and Theorem 2.6, the weak LO-proximity δ_F on X is compatible (and separated), and thus the LO-space (X, δ_F) is initial to $(X, f_i)_I$.

To prove the converse, suppose F is not a determinator on X. Since each member of F is continuous, δ_F must be coarser than the finest compatible LO-proximity on X. In fact the topology $\tau(\delta_F)$ induced by δ_F must be strictly coarser than the given topology, say τ, on X. So there is a subset A_0 of X such that A_0 is closed in (X, τ) but not closed in $(X, \tau(\delta_F))$. Let δ_c be the coarsest LO-proximity on X compatible with τ and set $\delta^* = \delta_c \vee \delta_F$. It is clear that no object except perhaps (X, δ^*) could be initial to (X, f_i, Y_i). But (X, δ^*) also fails to be initial to (X, f_i, Y_i) for the following reason: We can find a separated LO-space (Z, δ) and a continuous function $g: (Z, \tau(\delta)) \rightarrow (X, \tau)$ such that $g: (Z, \delta) \rightarrow (X, \delta_F)$ is p-continuous but $g: (Z, \delta) \rightarrow (X, \delta_c)$ is not p-continuous and hence $g: (Z, \delta) \rightarrow (X, \delta^*)$ is not p-continuous. The space (Z, δ) is constructed as follows:

Let A_0 be given the subspace topology inherited from the topology τ on X, and let Z be the disjoint topological union of A_0 and N where N is the discrete space of natural numbers. Since A_0 is not closed in the topology $\tau(\delta_F)$ on X there is a point $x_0 \notin A_0$ such that x_0 is in the closure of A_0 with regard to the topology $\tau(\delta_F)$ on X.

Define $g: Z \rightarrow X$ by $g(z) = z$ if $z \in A_0$ and $g(z) = x_0$ otherwise. Then $g: Z \rightarrow (X, \tau)$ is continuous. Let δ_g be the LO-proximity on Z defined by $(A, B) \in \delta_g$ if and only if $(gA, gB) \in \delta_F$ and let δ_1 be the coarsest compatible LO-proximity on Z. Set $\beta = \delta_g \cap \delta_1$, β is a LO-base on Z and $g: (Z, \delta(\beta)) \rightarrow (X, \delta_F)$ is p-continuous. Also if τ_0 is the topology on Z induced by $\delta(\beta)$ then $g: (Z, \tau_0) \rightarrow (X, \tau)$ is continuous. Now we complete the proof by showing that $g: (Z, \delta(\beta)) \rightarrow (X, \delta_c)$ is not p-continuous. Since each of the sets A_0 and N are infinite, it is clear that $(A_0, N) \in \beta$. Let $\{P_i: 1 \leqq i \leqq m\}$ and $\{N_j: 1 \leqq j \leqq n\}$ be any finite covers of A_0 and N respectively. Then there exists i_0, such that P_{i_0} is infinite and x_0 is in the $\tau(\delta_F)$-closure of $g(P_{i_0}) = P_{i_0}$. Also there is a j_0 such that N_{j_0} is infinite. Therefore $(P_{i_0}, N_{j_0}) \in \beta$ and hence $(A_0, N) \in \delta(\beta)$. This shows that $g: (Z, \delta(\beta)) \rightarrow (X, \delta_c)$ is not p-continuous and completes the proof.

An EF-space (X, δ) is called *p-stable* if and only if the p-class $\Pi(\delta)$ of uniformities on X compatible with δ has a finest member. In what follows, $\mathbf{U}$ denotes the category of uniform spaces and uniformly continuous maps.

(2.8) **Theorem.** *An EF-space (X, δ) is p-stable if and only if for each source $((X, \delta), f_i)_I$ in EF, there exists a source initial to it in* $\mathbf{U}$.

Proof. Suppose that δ is p-stable and let $\mathfrak{U}_1$ and $\mathfrak{U}_{\mathfrak{w}}$ be respectively the finest and the coarsest member of the p-class $\Pi(\delta)$ of uniformities compatible with δ. Let $\mathfrak{U}_F$ be the weak uniformity determined on X by $F = (f_i)_I$. Since each $f \in F$ is p-continuous, each member of F is uniformly continuous if the uniformity $\mathfrak{U}_1$ is taken on X. Therefore $\mathfrak{U}_F \subseteq \mathfrak{U}_1$. Let $\mathfrak{U}^* = \mathfrak{U}_F \vee \mathfrak{U}_{\mathfrak{w}}$. Then $\mathfrak{U}^*$ is compatible with δ and each member of F is uniformly continuous if we take the uniformity $\mathfrak{U}^*$ on X. In fact $\mathfrak{U}^*$ is the coarsest uniformity on X compatible with δ such that each function in F becomes uniformly continuous. Now we claim that $(X, \mathfrak{U}^*)$ is initial to the given source. Suppose $(Z, \mathfrak{U})$ is a uniform space such that (i) $g: (Z, \mathfrak{U}) \rightarrow (X, \delta)$

is p-continuous, and (ii) $f \cdot g$ is uniformly continuous for each f in F. The proximal continuity of g implies that $g: (Z, \mathfrak{U}) \rightarrow (X, \mathfrak{U}_{\mathfrak{D}})$ is uniformly continuous, and the fact that $f \cdot g$ is uniformly continuous for each f in F implies that $g: (Z, \mathfrak{U}) \rightarrow (X, \mathfrak{U}_F)$ is uniformly continuous. From these we conclude that $g: (Z, \mathfrak{U}) \rightarrow (X, \mathfrak{U}^*)$ is uniformly continuous.

Now to prove the converse, we suppose that (X, δ) is not p-stable. Let $\{\mathfrak{U}_\lambda: \lambda \in \Lambda\}$ be the p-class of uniformities of δ. For each $\lambda \in \Lambda$ take $I_\lambda: X \rightarrow (X, \mathfrak{U}_\lambda)$ defined by $I_\lambda(x) = x$ for all $x \in X$ and set $F = \{I_\lambda: \lambda \in \Lambda\}$. Since there is no finest member in the collection $\{\mathfrak{U}_\lambda: \lambda \in \Lambda\}$ it follows that (X, δ) cannot be lifted to an initial source.

It would be interesting to have a characterization of those sources which the functor $(\mathbf{U} - \mathbf{T}_{3\frac{1}{2}})$ lifts to initial sources.

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