PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 3, September 1974

THE MINIMUM MODULUS OF POLYNOMIALS

E. BELLER AND D. J. NEWMAN[1]

ABSTRACT. In answer to a problem of Erdös and Littlewood we produce an nth degree polynomial, $P(z)$, with coefficients bounded by 1 satisfying $|P(z)| > C\sqrt{n}$ for all z on $|z| = 1$ (C is a positive absolute constant).

Littlewood and Erdös, independently, asked whether polynomials, $P(z)$, of degree n could exist having all coefficients bounded by 1 and satisfying $\min_{|z|=1} |P(z)| \geq C\sqrt{n}$ (C a fixed positive constant).

Clunie [2] gave a very ingenious construction of a polynomial purporting to do this job, but it was based on a result of Littlewood's [3] which later proved erroneous. Littlewood claimed that, as $r \to 1^-$,

$$\min_{|z|=r} \left| \sum_{n=1}^{\infty} n^{in} z^n \right| = \Omega(1-r)^{-1/2},$$

but his reasoning had a flaw which was discovered by Erdös and Carroll. Indeed a careful examination of his method shows the very opposite, that

$$\min_{|z|=r} \left| \sum_{n=1}^{\infty} n^{in} z^n \right| = o(1-r)^{-1/2}.$$

In this note we give an extremely simple construction of a polynomial which does have the desired properties.

Consider the function $f(\theta)$ defined as $\exp(in\delta\theta^2)$ in $[-\pi, \pi]$ and extended to have period 2π. (δ is a small but fixed positive number.) Write $K = [n/2]$, let $t(\theta)$ be the Kth Cesàro partial sum of the Fourier series of $f(\theta)$, and finally set $P(e^{i\theta}) = \sqrt{n\delta}\, e^{iK\theta} t(\theta)$.

Clearly $P(z)$ is a polynomial of degree $\leq n$ and we will prove

(1) The coefficients of $P(z)$ are all bounded by 1.

(2) $|P(z)| \geq \sqrt{n\delta}(1 - 40\,\delta \log \delta^{-1})$ for all z on $|z| = 1$.

Received by the editors December 10, 1973.

AMS (MOS) subject classifications (1970). Primary 26A75, 26A82, 30A06.

Key words and phrases. Polynomials, minimum modulus, Fourier series.

[1] The second author was supported in part by AFOSR 72-2380A.

To prove (1) we start with the exact formula for the coefficient of z^j, namely

$$\left(1-\frac{|j-K|}{K+1}\right)\frac{\sqrt{n\delta}}{2\pi}\int_{-\pi}^{\pi}\exp(i(n\delta\theta^2+(K-j)\theta))\,d\theta\,.$$

Since $n\delta\theta^2+(K-j)\theta$ has second derivative equal to $2n\delta$ we may apply Lemma 4.4, p. 61 of [4] and obtain the bound

$$\left|\int_{-\pi}^{\pi}\exp\,(i(n\delta\theta^2+(K-j)\theta))\,d\theta\right|\le\frac{8}{\sqrt{2n\delta}}\,.$$

Thus our coefficient is bounded by

$$\left(1-\frac{|j-K|}{K+1}\right)\frac{\sqrt{n\delta}}{2\pi}\cdot\frac{8}{\sqrt{2n\delta}}<1$$

as required.

To prove (2) we turn to Fejér's formula

$$t(\theta)-f(\theta)=\frac{1}{2\pi(K+1)}\int_{-\pi}^{\pi}(f(\theta+u)-f(\theta))\frac{\sin^2((K+1)/2)}{\sin^2(u/2)}\,du\,.$$

Now $f(\theta)$ has derivative bounded by $2\pi n\delta$ so that $|f(\theta+u)-f(\theta)|\le 2\pi n\delta|u|$. Also, of course, $|f(\theta+u)-f(u)|\le 2$.

Using this, together with the elementary inequalities

$$\frac{\sin^2((K+1)u/2)}{\sin^2(u/2)}\le(K+1)^2\quad\text{and}\quad\frac{\sin^2((K+1)u/2)}{\sin^2(u/2)}\le\frac{1}{\sin^2(u/2)}\le\frac{\pi^2}{u^2},$$

we obtain

$$(K+1)|t(\theta)-f(\theta)|$$

$$\le\int_0^{\pi/(K+1)}2n\delta u\cdot(K+1)^2\,du+\int_{\pi/(K+1)}^{1/\pi n\delta}2n\delta u\cdot\frac{\pi^2}{u^2}du+\int_{1/\pi n\delta}^{\pi}\frac{2\pi}{u^2}\,du$$

$$=2\pi^2n\delta\log\frac{(K+1)e^{3/2}}{\pi^2n\delta}-2\le 2\pi^2n\delta\log\frac{e^{3/2}}{\pi^2\delta}-1<20n\delta\log\frac{1}{\delta}\,.$$

Since $K+1>n/2$ we may thereby conclude that

$$|t(\theta)|\ge 1-|t(\theta)-f(\theta)|>1-40\delta\log\delta^{-1}$$

and (2) follows immediately.

There is an extra dividend issuing from this construction. If we simply set $\delta=1/400$, for example, then we do obtain a polynomial which does the Littlewood-Clunie job. Notice, however, that $t(\theta)$ is automatically bounded by 1, the bound for $f(\theta)$, so that we obtain the upper bound $|P(z)|\le\sqrt{n\delta}$.

Hence if we make δ very much smaller, our polynomial will have the additional property that

$$\operatorname{Max}|P(z)| \leq (1+\epsilon) \min |P(z)|.$$

The constant we obtain is quite a poor one, but it can be improved using the method given in [1]. Indeed we can produce a $P(z)$ such that $|P(z)| \geq .395 \sqrt{n}$ for large n.

REFERENCES

1. E. Beller and D. J. Newman, *An l_1 extremal problem for polynomials*, Proc. Amer. Math. Soc. 29 (1971), 474–481. MR 43 #6407.

2. J. Clunie, *The minimum modulus of a polynomial on the unit circle*, Quart. J. Math. Oxford Ser. (2) 10 (1959), 95–98. MR 21 #5005.

3. J. E. Littlewood, *Lectures on the theory of functions*, Oxford Univ. Press, Oxford, 1944, pp. 100–102. MR 6, 261.

4. E. C. Titchmarsh, *Theory of the Riemann zeta function*, Clarendon Press, Oxford, 1951. MR 13, 741.

DEPARTMENT OF MATHEMATICS, BAR ILAN UNIVERSITY, RAMAT GAN, ISRAEL

DEPARTMENT OF MATHEMATICS, YESHIVA UNIVERSITY, NEW YORK, NEW YORK 10033