PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 3, September 1974

REDUCING SUBSPACES OF CONTRACTIONS WITH NO ISOMETRIC PART

JAMES GUYKER[1]

ABSTRACT. Let T be a contraction on a Hilbert space H and suppose that there is no nonzero vector f in H such that $\|T^n f\| = \|f\|$ for every $n = 1, 2, \cdots$. In this paper, the reducing subspaces of T are characterized in terms of the range of $1 - T^* T$. As a corollary, it is shown that T is irreducible if $1 - T^* T$ has 1-dimensional range. In particular, if U is the simple unilateral shift, then the restriction of U^* to any invariant subspace for U^* is irreducible.

Let T be a contraction on a Hilbert space H. We recall that a (closed) subspace M of H reduces T if M is invariant under both T and T^*. If the only subspaces that reduce T are $\{0\}$ and H itself, then T is said to be irreducible.

The contraction T has no isometric part if there is no nonzero vector f in H such that $\|T^n f\| = \|f\|$ for every $n = 1, 2, \cdots$. The structure of the reducing subspaces for the adjoint of a unilateral shift is well known [2, Lemma 3.2, p. 724], [3, Theorem 1, p. 105]. In the present paper, the structure is obtained for arbitrary contractions with no isometric part.

If E is a subset of H, then $\bigvee E$ will denote the closed span of E. For subspaces M and N of H such that $N \subset M$, $M \ominus N$ will denote the orthogonal complement of N in M.

Theorem. *Let T be a contraction on a Hilbert space H and suppose that T has no isometric part. Let K be the closure of the range of $1 - T^*T$. A subspace M of H reduces T if and only if $M = \bigvee\{T^{*^n} f \colon f \in S, n \geq 0\}$ for some unique subspace S of K which is invariant under $(1 - T^*T)T^m T^{*^n}$ for every $m, n = 0, 1, 2, \cdots$. In this case*

Received by the editors September 9, 1970 and, in revised form, January 16, 1971.

AMS (MOS) subject classifications (1970). Primary 47A15; Secondary 47A65.

Key words and phrases. Contraction with no isometric part, reducing subspace, irreducible contraction, unilateral shift.

[1] This work is part of the author's doctoral dissertation written under the direction of Professor David Trutt at Lehigh University.

$$H \ominus M = \bigvee \{T^{*n} f\colon f \in K \ominus S,\ n \geq 0\}.$$

Proof. Suppose that M reduces T and let S be the closure of $(1 - T^*T)M$ in H. Clearly $\bigvee\{T^{*n} f\colon f \in S, n \geq 0\} \subset M$, and if g is in $M \ominus \bigvee\{T^{*n} f\colon f \in S, n \geq 0\}$, then

$$\langle g, T^{*n}(1 - T^*T)T^n g\rangle = 0$$

for every $n = 0, 1, 2, \cdots$. Hence $\|T^{n+1}g\| = \|T^n g\|$ for every $n = 0, 1, 2, \cdots$. Since T has no isometric part, $g = 0$. Therefore

$$M = \bigvee \{T^{*n} f\colon f \in S,\ n \geq 0\}.$$

Since M reduces T, we have that $K \ominus S$ is the closure in H of $(1 - T^*T)(H \ominus M)$ and therefore $H \ominus M = \bigvee\{T^{*n} f\colon f \in K \ominus S, n \geq 0\}$.

Conversely, suppose that $M = \bigvee\{T^{*n} f\colon f \in S', n \geq 0\}$ where S' is a subspace of K which is invariant under $(1 - T^*T)T^m T^{*n}$ for every $m, n = 0, 1, 2, \cdots$. Let

$$N = \{g \in H\colon (1 - T^*T)T^m g \in S', \forall m = 0, 1, 2, \cdots\}.$$

Clearly $N \supset M$. Let g belong to N. Since $(1 - T^*T)T^n g$ is in S', we have that $T^{*n}(1 - T^*T)T^n g$ is in M for every $n = 0, 1, 2, \cdots$. It follows as above that $N = M$ and hence that M reduces T.

Let S be the closure of $(1 - T^*T)M$ in H. Clearly $S \subset S'$ since $(1 - T^*T)T^{*n} f$ is in S' for every f in S' and for every $n = 0, 1, 2, \cdots$. As above, $H \ominus M = \bigvee\{T^{*n} f\colon f \in K \ominus S, n \geq 0\}$. It follows that $S' \ominus S$ is contained in both M and $H \ominus M$, and consequently $S' = S$.

Corollary 1. *If T is a contraction with no isometric part and $1 - T^*T$ has 1-dimensional range, then T is irreducible.*

Proof. In the Theorem, let K be 1-dimensional. Since the only subspaces S of K are therefore $\{0\}$ and K itself, it follows that the only subspaces M of H that reduce T are $\{0\}$ and H itself.

A basic result is that the restriction of the simple unilateral shift U to any invariant subspace for U is irreducible [3]. Corollary 1 immediately implies

Corollary 2. *The restriction of U^* to any invariant subspace for U^* is irreducible.*

Remark. The above theorem was obtained by proving special cases with the use of the canonical model of de Branges and Rovnyak [1].

REFERENCES

1. L. de Branges and J. Rovnyak, *Canonical models in quantum scattering theory*, Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U. S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965), Wiley, New York, 1966, pp. 295–391. MR **39** #6109.

2. A. Brown, *On a class of operators*, Proc. Amer. Math. Soc. **4** (1953), 723–728. MR **15**, 538.

3. P. R. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. **208** (1961), 102–112. MR **27** #2868.

DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY, BETHLEHEM, PENNSYLVANIA 18015

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, COLLEGE AT BUFFALO, BUFFALO, NEW YORK 14222