PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 2, November 1974

# EVERY DIRECTION A JULIA DIRECTION

BRYAN E. CAIN

ABSTRACT. Let $f(z) = \exp(\cosh z)$. If $N$ is any $\epsilon$-neighborhood of any ray through the origin with slope $m \neq 0, \infty$ then $f^{-1}(w) \cap N$ is infinite if $w \neq 0$.

Let $J[f]$ denote the set of Julia directions of the entire function $f$. That is, $\theta \in J[f]$ if, in every sector $\alpha < \arg z < \beta$ such that $\alpha < \theta < \beta$, $f$ assumes every complex value, with at most one exception, infinitely often. Using infinite products Julia [2] has constructed an entire function for which every direction is a Julia direction. The example which follows is more elementary.

The closed annulus $\{1/n \leq |z| \leq n\}$ will be denoted $A_n$ for $n = 1, 2, \cdots$. If $a, b, \delta > 0$ the closed rectangle $\{x + iy: -a \leq x \leq a: b - \delta \leq y \leq b + \delta\}$ will be denoted by $R_\delta(a, b)$.

**Theorem.** *If* $f(z) = \exp(\cosh z)$ *then* $J[f] = \mathbf{R}$.

**Proof.** The relations $f(\bar{z}) = (f(z))^-$ and $f(z) = f(-z)$ imply that if $\theta \in J[f]$ then $-\theta, \theta \pm \pi \in J[f]$. Consequently we can finish the proof by showing that $(0, \pi/2) \subset J[f]$ because the set of Julia directions is clearly always closed.

Now suppose that $\theta, \alpha, \beta$ are given and that they satisfy $0 < \theta < \pi/2$ and $\alpha < \theta < \beta$. They will be fixed for the rest of the proof.

Let $S = \{x + iy: mx - \epsilon < y < mx + \epsilon\}$ where $m = \tan\theta$ and $0 < \epsilon < \pi$. The function $z \to \frac{1}{2}\exp(z)$ maps the strip $S$ onto a ribbon which starts at the origin, wraps around it infinitely often, and spirals out to infinity. This ribbon has $\gamma_-$ for its inside boundary and $\gamma_+$ for its outside boundary where

$$\gamma_\pm(y) = \tfrac{1}{2}\exp[(y \pm \epsilon)/m + iy]$$

for all $y$. The width of the ribbon $|\gamma_+(y) - \gamma_-(y)|$, which is measured along a ray from the origin, is an unbounded increasing function of $y$. The ribbon does not overlap itself because $|\gamma_-(y + 2\pi)| - |\gamma_+(y)| > 0$ provided $\epsilon < \pi$.

Received by the editors August 30, 1973.
*AMS (MOS) subject classifications* (1970). Primary 30A70.

Let $P_n$ denote the open parallelogram which is the intersection of the strip $\{x + iy: 2\pi n < y < 2\pi n + \pi\}$ with the strip $S$. Note that for all $n$ larger than some $n_0$ the sector $\alpha < \arg z < \beta$ will contain $P_n$, and that the $P_n$'s are disjoint. Thus the proof can be finished by showing that if $w \neq 0$ then $w \in f(P_n)$ for infinitely many $n$.

Let $ic_n$ be the midpoint of the interval in which $\frac{1}{2}\exp(P_n)$ meets the imaginary axis. Since $\frac{1}{2}\exp(P_n)$ is the interior of one component of the portion of the ribbon lying in the half plane $\operatorname{Im} z > 0$, we know that $c_n > 0$.

**Lemma.** *If $a > 0$ then $R_\pi(a, c_k) \subseteq \cosh(P_k)$ for infinitely many values of $k$.*

Using this lemma we let $U_n = P_k$, where $k > n_0$ is chosen so large that $R_\pi(\log n, c_k) \subseteq \cosh(P_k)$ and $P_k$ is disjoint from $U_1, \cdots, U_{n-1}$. Then $f(U_n) \supseteq \exp(R_\pi(\log n, c_k)) = A_n$, and, since every $w \neq 0$ lies in infinitely many $A_n$, $f^{-1}(w) \cap \{\alpha < \arg z < \beta\}$ is infinite. Thus $\theta \in J[f]$. $\square$

**Remark.** This proof actually shows that if $\theta \not\equiv 0$ or $\pi/2 \pmod{\pi}$ and if $N$ is any $\epsilon$-neighborhood of a ray from the origin through $e^{i\theta}$, then $f^{-1}(w) \cap N$ is infinite if $w \neq 0$. Had we merely wished to prove the Theorem we could have replaced the $P_n$'s with a sequence of disjoint rectangles $R_n = \{x + iy: a_n < x < b_n, 2\pi n < y < 2\pi n + \pi\}$ which lie in $\alpha < \arg z < \beta$ and for which the sequences $a_n$ and $b_n - a_n$ approach $\infty$. Then $\frac{1}{2}\exp(R_n)$ is the intersection of the half plane $\operatorname{Im} z > 0$ with the annulus $\exp(a_n) < |z| < \exp(b_n)$. Since the width $\exp(a_n) - \exp(b_n)$ of the annulus approaches $\infty$, it is geometrically clear that if $a > 0$ and if $c_n = \frac{1}{2}(\exp(a_n) + \exp(b_n))$, there will exist infinitely many $n$'s for which the rectangle $R_\pi(a, c_n)$ lies inside $\frac{1}{2}\exp(R_n)$. But when $a_n$ is large the boundary of $\cosh(R_n)$ will stay very close to the boundary of $\frac{1}{2}\exp(R_n)$ because then $\frac{1}{2}\exp(-R_n)$ must lie within a tiny neighborhood of 0. Thus $R_\pi(a, c_n) \subseteq \cosh(R_n)$ for infinitely many $n$'s. Using this version of the Lemma, the Theorem can be proved by replacing the parallelograms $P_n$ in the proof above with the rectangles $R_n$.

**Proof of the Lemma.** Suppose that $\delta > 0$ and let $b_n$ be the largest number such that the interior of $R_\delta(b_n, c_n)$ is contained in $\frac{1}{2}\exp(P_n)$. (Once the width of the ribbon exceeds $2\delta$, $b_n$ will be positive.) Then some vertex of $R_\delta(b_n, c_n)$ lies on $\gamma_+$ or $\gamma_-$, and we shall show that this implies the unboundedness of $\{b_n\}$. Since $|\gamma_\pm(y)|$ are increasing functions of $y$ there are just two cases: (1) the northeast vertex $b_n + i(c_n + \delta)$ lies on $\gamma_+$, or (2) the southwest vertex $-b_n + i(c_n - \delta)$ lies on $\gamma_-$.

When case (1) holds we have

(A) $\quad b_n + i(c_n + \delta) = \gamma_+(2\pi n + \phi_n)$ where $\phi_n = \tan^{-1}[(c_n + \delta)/b_n]$.

Assuming that $b_n$ is bounded implies that $\phi_n \longrightarrow \pi/2$ because $c_n \longrightarrow \infty$. Now we divide equation (A) by $\gamma_+(2\pi n + \pi/2)$. Since

$$c_n = \frac{1}{2i}\left(\gamma_+\left(2\pi n + \frac{\pi}{2}\right) + \gamma_-\left(2\pi n + \frac{\pi}{2}\right)\right) \quad \text{and since} \quad \frac{\gamma_-(y)}{\gamma_+(y)} = \exp\left[\frac{-2\epsilon}{m}\right]$$

the left side becomes

$$\left[\frac{b_n + i\delta}{\gamma_+(2\pi n + \pi/2)}\right] + \frac{1}{2}\left(1 + \exp\left[\frac{-2\epsilon}{m}\right]\right).$$

The right side becomes $\exp[(1/m + i)(\phi_n - \pi/2)]$ and if case (1) obtains for infinitely many $n$ we can equate the limit as $n \longrightarrow \infty$ of each side and produce the contradiction $\frac{1}{2}(1 + \exp[-2\epsilon/m]) = 1$.

Case (2) gives

(B) $\quad -b_n + i(c_n - \delta) = \gamma_-(2\pi n + \pi - \psi_n)$ where $\psi_n = \tan^{-1}[(c_n - \delta)/b_n]$.

If $b_n$ is bounded and (B) holds infinitely often, then dividing by $\gamma_-(2\pi n + \pi/2)$ and letting $n$ approach $\infty$ makes $\psi_n \longrightarrow \pi/2$ and gives the contradiction $\frac{1}{2}(\exp(2\epsilon/m) + 1) = 1$.

This proves that $b_n$ is unbounded. Thus, in particular, if $\delta = \pi + 1$ and $b = a + 1$ the inclusion $R_\pi(a, c_n) \subset R_\delta(b, c_n) \subset \frac{1}{2}\exp(P_n)$ holds for infinitely many values of $n$. Then when $n$ is large enough the set $\frac{1}{2}\exp(-P_n)$ will lie in a neighborhood of 0 so small that $\cosh(P_n)$ very nearly contains $R_\delta(b, c_n)$ and certainly contains $R_\pi(a, c_n)$. $\square$

**Acknowledgement.** We are indebted to Peter Colwell for provocative discussions and to Richard Tondra for improving our example.

## REFERENCES

1. E. Hille, *Analytic function theory*. Vol. II, Introductions to Higher Math., Ginn, New York, 1962. MR 34 #1490.

2. G. Julia, *Sur quelques propriétés nouvelles des fonctions entières ou méromorphes* (liere Mémoire), Ann. Sci. École Norm. Sup. 36 (1919), 93–125.

3. T. Zinno, *Some properties of Julia's exceptional functions and an example of Julia's exceptional functions with Julia's direction*, Ann. Acad. Sci. Fenn. Ser. A I, No. 464 (1970), 12 pp. MR 43 #6414.

DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA 50010

*Current address*: Mathematiches Institut der Technishen Universität München, 8 München 2, Arcisstrasse 21, Postfach 202 420, Federal Republic of Germany