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CHARACTERIZATION OF THE FLIP OPERATOR[1]

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ABSTRACT. The flip operator F on $L_p([0, 1])$, defined by $F(f)(t) = f(1-t)$, for f in $L_p([0, 1])$ is characterized up to isometric transformation by means of its induced σ-isomorphism on the Borel sets of $[0, 1]$.

Introduction. The bounded operator F on $L_p([0, 1])$ defined by $F(f)(t) = f(1-t)$, for f in $L_p([0, 1])$, is called the flip operator. It is a surjective isometry and $F^2 = I$, the identity operator. There is a certain σ-isomorphism on the Borel sets of $[0, 1]$ which is induced by F. It is our purpose to show that these properties characterize F up to isometric transformation.

1. **Preliminaries.** We shall assume from the outset that all measure spaces are σ-finite and nonatomic.

Let (X, Σ, μ) be a measure space and let N_μ be $\{\sigma \in \Sigma | \mu(\sigma) = 0\}$. We say that two elements σ and τ in Σ are equal almost everywhere if $\sigma \Delta \tau$ is in N_μ. This is an equivalence relation on Σ and we denote the equivalence class of $\sigma \in \Sigma$ by $[\sigma]$. The collection of all such classes is denoted by Σ'. We define all the usual set operations on Σ' in terms of these operations on representatives in Σ as in [3, Chapter 14, §2].

Definition 1.1. Let (X, Σ, μ) and (Y, Φ, ν) be measure spaces. A map $\Gamma: \Sigma' \to \Phi'$ is a *σ-isomorphism* if:

(i) Γ is a bijection;

(ii) $\Gamma([\sigma] \setminus [\tau]) = \Gamma([\sigma]) \setminus \Gamma([\tau])$ for $[\sigma]$ and $[\tau]$ in Σ';

(iii) if $\{[\sigma_i]\}$ is a sequence in Σ', then $\Gamma(\bigcup_{i=1}^\infty [\sigma_i]) = \bigcup_{i=1}^\infty \Gamma([\sigma_i])$.

Definition 1.2. Let $f \in L_p(X, \Sigma, \mu)$. Then $\{x: |f(x)| > 0\}$ is called the *support of f* (written supp(f)).

Remark 1.1. It has been shown that if $J: L_p(X, \Sigma, \mu) \to L_p(Y, \Phi, \nu)$,

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is a surjective isometry, where $1 \leq p < \infty$ and $p \neq 2$, then J induces a map $\Gamma: \Sigma' \to \Phi'$ in the following manner. Let $\sigma \in \Sigma$ and suppose $f \in L_p(X, \Sigma, \mu)$ is such that $\mathrm{supp}(f) = \sigma$ a.e. μ. We set $\Gamma([\sigma]) = [\mathrm{supp}(J(f))]$. Then Γ is well defined and a σ-isomorphism (see [2] and [4, Theorem 1.2]). We say that J *induces* Γ.

Let X and Y be topological spaces with Borel sets $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$ respectively. Let μ and ν be σ-finite measures on $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$ respectively.

Definition 1.3. A bijective map $\theta: X \setminus X_0 \to Y \setminus Y_0$, where $\mu(X_0) = \nu(Y_0) = 0$, is a *Borel equivalence* if:

(i) θ and θ^{-1} are measurable;

(ii) if $\sigma \in \mathfrak{B}(X)|_{X \setminus X_0}$ then $\mu(\sigma) = 0$ if and only if $\nu(\theta(\sigma)) = 0$.

Remark 1.2. Clearly every Borel equivalence θ induces a σ-isomorphism $\Theta: \mathfrak{B}'(X) \to \mathfrak{B}'(Y)$ via $\Theta([\sigma]) = [\theta(\sigma)]$ for $\sigma \in \mathfrak{B}(X)$.

If X and Y are complete separable metric spaces, and if $\Gamma: \mathfrak{B}'(X) \to \mathfrak{B}'(Y)$ is a σ-isomorphism, then Γ is induced by a Borel equivalence γ (see, e.g., [3, Corollary 12, p. 272]).

In the sequel, we shall denote $L_p(X, \Sigma, \mu)$ by $L_p(\mu)$, $1 \leq p < \infty$, and by identifying sets in Σ equal almost everywhere, we write $\sigma \in \Sigma$ to represent $[\sigma]$ in Σ'. The usual Borel measure space $([0, 1], \mathfrak{B}([0, 1]), \lambda)$ will be abbreviated by $[0, 1]$.

2. Properties of the flip operator.

Definition 2.1. A function f defined and measurable on $[0, 1]$ is *symmetric with respect to* ½ *a.e.* if $f(t) = f(1 - t)$ for almost all $t \in [0, 1]$.

A set $\beta \subset [0, 1]$ is *symmetric with respect to* ½ *a.e.* if $t \in \beta$ if and only if $(1 - t) \in \beta$ for almost all $t \in [0, 1]$. (It follows that if β is symmetric with respect to ½ a.e., then $[0, 1] \setminus \beta$ is also symmetric with respect to ½ a.e.)

Theorem 2.1. *The operator F is the only operator T on $L_p([0, 1])$ with the properties*:

(i) *T is a surjective isometry that canonically induces a σ-isomorphism Γ on $\mathfrak{B}'([0, 1])$*;

(ii) *$T(\chi(\beta)) = \chi(\beta)$ if and only if $\chi(\beta)$ is a characteristic function symmetric with respect to* ½ *a.e.*

Proof. Clearly F has properties (i) and (ii).

Suppose that an operator T satisfies (i) and (ii). Then there exist β_0

and β_1 in $\mathfrak{B}([0, 1])$, with $\lambda(\beta_0) = \lambda(\beta_1) = 0$ and a Borel equivalence θ: $[0, 1]\setminus\beta_0 \to [0, 1]\setminus\beta_1$ such that for $f \in L_p([0, 1])$, it follows that $T(f)(t) = q(t)f(\theta(t))$ a.e., where $q(t) \in L_p([0, 1])$ (see [4, Remark 2.2 and Theorem 1.5]). Thus since $T(\chi([0, 1])) = \chi([0, 1])$, we see that $T(f) = f \circ \theta$ a.e., and an elementary argument shows that $T(k) = k$ whenever $F(k) = k$, for $k \in L_p([0, 1])$.

Consider the symmetric function defined by

$$\begin{aligned} f(t) &= t, & t &\in [0, \tfrac{1}{2}], \\ &= 1 - t, & t &\in [\tfrac{1}{2}, 1]. \end{aligned}$$

Then $T(f)(t) = f(t) = (f \circ \theta)(t)$ a.e. on $[0, 1]\setminus\beta_0$. It follows that $\theta(t) = t$ or $\theta(t) = 1 - t$ for almost all $t \in [0, 1]\setminus\beta_0$.

Let $\gamma = \{t \in [0, 1]\setminus\beta_0 | \theta(t) = t\}$. If $\lambda(\gamma) > 0$, then we have $T(\chi(\gamma)) = \chi(\gamma)$ since θ is injective. Thus γ is symmetric with respect to $\frac{1}{2}$ a.e. Let $\delta = \gamma \cap [0, \frac{1}{2}]$. It follows that $T(\chi(\delta)) = \chi(\delta)$. Thus we conclude that $\lambda(\delta) = 0$. This is a contradiction. Thus $\theta(t) = 1 - t$ a.e. on $[0, 1]\setminus\beta_0$. Hence for any $k \in L_p([0, 1])$, we see that $T(k)(t) = k(1 - t) = F(k)(t)$ a.e.

The flip operator F on $L_p([0, 1])$ induces a σ-isomorphism Λ: $\mathfrak{B}'([0, 1]) \to \mathfrak{B}'([0, 1])$ induced by the Borel equivalence ϕ: $[0, 1] \to [0, 1]$ defined by $\phi(t) = 1 - t$. We observe that Λ has the property that $\Lambda([0, \frac{1}{2}]) = [\frac{1}{2}, 1]$.

Remark 2.1. If Γ_I: $\mathfrak{B}'([0, 1]) \to \mathfrak{B}'([0, 1])$ is the identity σ-isomorphism, then it is induced solely by ι, the identity Borel equivalence. For consider the identity operator I on $L_p([0, 1])$. Suppose there exists θ, a Borel equivalence, such that θ induces Γ_I. Then for any $f \in L_p([0, 1])$, we have $(f \circ \theta)(t) = I(f)(t) = f(t)$ a.e. In particular, choose $f(t) = t$. Then we see that $\theta(t) = t$ a.e., i.e., $\theta = \iota$ a.e. As a consequence, if $(X, \mathfrak{B}(X), \mu)$ and $(Y, \mathfrak{B}(Y), \nu)$ are two Borel measure spaces with X and Y complete separable metric spaces, then any σ-isomorphism Γ: $\mathfrak{B}'(X) \to \mathfrak{B}'(Y)$ is induced by a unique Borel equivalence.

Definition 2.2. A bounded operator R on $L_p(\mu)$ is *isometrically equivalent* to a bounded operator S on $L_p(\nu)$ if there exists a surjective isometry J: $L_p(\mu) \to L_p(\nu)$ such that $R = J^{-1}SJ$.

Lemma 2.1. *Let $(X, \mathfrak{B}(X), \mu)$ be a Borel nonatomic measure space, where X is a complete separable metric space. A bounded operator T on $L_p(\mu)$, $p \neq 2$, is isometrically equivalent to F on $L_p([0, 1])$ if and only if:*

(i) *T is a surjective isometry such that $T^2 = I$;*

(ii) *there exists $\beta \in \mathfrak{B}(X)$ such that $\Gamma(\beta) = X\setminus\beta$ where Γ is the σ-isomorphism induced by T.*

Proof. If T is isometrically equivalent to F, then clearly (i) and (ii) hold. So suppose T is such that (i) and (ii) hold. The measure space $(\beta, \mathcal{B}(X)|_{\beta}, \mu|_{\beta})$ is a separable nonatomic measure space Borel equivalent to $([0, \tfrac{1}{2}], \mathcal{B}([0, \tfrac{1}{2}]), \lambda|_{[0,\frac{1}{2}]})$ (see, e.g., [1, Theorem C, p. 173] and [3, Corollary 12, p. 272]). So there exists $\delta_0 \in \mathcal{B}([0, \tfrac{1}{2}])$, such that $\{\tfrac{1}{2}\} \subset \delta_0$ and $\lambda(\delta_0) = \mu(\beta_0) = 0$, and a Borel equivalence τ: $[0, \tfrac{1}{2}]\setminus\delta_0 \to \beta\setminus\beta_0$. We extend τ to a Borel equivalence $\bar{\tau}$: $[0, 1]\setminus\delta_1 \to X\setminus\beta_1$, where $\delta_1 \in \mathcal{B}([0, 1])$, $\beta_1 \in \mathcal{B}(X)$, and $\lambda(\delta_1) = \mu(\beta_1) = 0$, in the following manner. The σ-isomorphism Γ induced by T is also induced by a Borel equivalence θ: $X\setminus\beta_2 \to X\setminus\beta_3$ where $\mu(\beta_2) = \mu(\beta_3) = 0$. Let $\beta_4 = \beta_2 \cup \theta^{-1}(\beta_2)$. Then $\theta|_{X\setminus\beta_4}$ is a Borel equivalence and the composition $\theta \circ (\theta|_{X\setminus\beta_4})$: $X\setminus\beta_4 \to X$ is a well-defined measurable map which induces the identity σ-isomorphism Γ_I on $\mathcal{B}'(X)$ since $T^2 = I$. However Γ_I is also induced by ι, the identity Borel equivalence on X. Thus there exists $\beta_5 \in \mathcal{B}(X)$ such that $\beta_5 \supset \beta_4$, $\mu(\beta_5) = 0$, and $\theta^2 = \iota$ on $X\setminus\beta_5$.

Let δ_2 be $\{t \in (\tfrac{1}{2}, 1] | (1 - t) \in \delta_0 \cup \tau^{-1}(\beta_5)\}$. We see that $\lambda(\delta_2) = 0$ since $\chi(\delta_2) = F(\chi(\delta_0 \cup \tau^{-1}(\beta_5)))$.

Let δ_1 be $\delta_0 \cup \delta_2$. Then if ϕ: $[0, 1] \to [0, 1]$ is the mapping defined by $\phi(t) = 1 - t$, we define

$$\begin{aligned} \bar{\tau}(t) &= \tau(t), && t \in [0, \tfrac{1}{2}]\setminus\delta_1, \\ &= \theta \circ \tau \circ \phi(t), && t \in [\tfrac{1}{2}, 1]\setminus\delta_1. \end{aligned}$$

Since $\tau([0, \tfrac{1}{2}]\setminus\delta_1) = \beta$ a.e. μ and since θ induces Γ, we see by (ii) that $\bar{\tau}((\tfrac{1}{2}, 1]\setminus\delta_1) = X\setminus\beta$ a.e. μ. Also $\bar{\tau}$ is a Borel equivalence as a consequence of the fact that τ, θ, and ϕ are Borel equivalences.

We observe that if t is in $[0, \tfrac{1}{2}]\setminus\delta_1$, then there exists $x_t \in \beta$ such that $\tau(t) = x_t$.

Let δ_3 be $\delta_1 \cup \tau^{-1}(\beta_5) \cup \{t | (1 - t) \in \delta_1\}$. Then we have $\lambda(\delta_3) = 0$. We conclude that $\phi(t) = \bar{\tau}^{-1} \circ \theta \circ \bar{\tau}(t)$ for $t \in [0, 1]\setminus\delta_3$. For if $t \in [0, \tfrac{1}{2}]\setminus\delta_3$, it follows that

$$\begin{aligned} \bar{\tau}^{-1} \circ \theta \circ \bar{\tau}(t) &= \bar{\tau}^{-1} \circ \theta \circ \tau(t) = \bar{\tau}^{-1} \circ \theta(x_t) \\ &= \bar{\tau}^{-1} \circ \bar{\tau}(1 - t) = 1 - t. \end{aligned}$$

If t is in $(\tfrac{1}{2}, 1]\setminus\delta_3$, then we have

$$\begin{aligned} \bar{\tau}^{-1} \circ \theta \circ \bar{\tau}(t) &= \bar{\tau}^{-1} \circ \theta \circ \theta \circ \tau \circ \phi(t) = \bar{\tau}^{-1} \circ \tau \circ \phi(t) \\ &= \bar{\tau}^{-1} \circ \tau(1 - t) = \bar{\tau}^{-1} \circ \bar{\tau}(1 - t) = 1 - t. \end{aligned}$$

We see that ϕ induces the same σ-isomorphism Λ on $\mathcal{B}'([0, 1])$ as is induced by F. Let $\Xi: \mathcal{B}'([0, 1]) \to \mathcal{B}'(X)$ be the σ-isomorphism induced by $\bar{\tau}$. Then there exists a surjective isometry $J_{\Xi} : L_p([0, 1]) \to L_p(\mu)$ such that $TJ_{\Xi} = J_{\Xi} R$ for some bounded operator R on $L_p([0, 1])$. It follows that R is a surjective isometry inducing Λ. For suppose f is in $L_p([0, 1])$ and $\operatorname{supp}(f) = \delta \in \mathcal{B}([0, 1])$. Then we have

$$\operatorname{supp}(J_{\Xi}^{-1} T J_{\Xi}(f)) = \bar{\tau}^{-1} \circ \theta \circ \bar{\tau}(\delta) = \phi(\delta) = \Lambda(\delta) \quad \text{a.e. } \lambda.$$

Thus there exists $h \in L_\infty([0, 1])$, where $|h| = 1$ a.e. λ, such that $R = M_h F$, where M_h on $L_p([0, 1])$ is the operator defined by $M_h(f) = h \cdot f$ (see, e.g., [4, Theorem 1.5]). By (i) we see that $R^2 = M_h F M_h F = I$ and it follows that $h(t)h(1-t) = 1$ a.e. λ.

Define $k \in L_\infty([0, 1])$ by

$$\begin{aligned} k(t) &= 1, & t &\in [0, \tfrac{1}{2}], \\ &= h(1-t), & t &\in (\tfrac{1}{2}, 1]. \end{aligned}$$

Then the operator M_k on $L_p([0, 1])$, defined by $M_k(f) = k \cdot f$, is a surjective isometry since $|k| = 1$ a.e., and $FM_k = M_k R$. We conclude that $FM_k J_{\Xi}^{-1} = M_k J_{\Xi}^{-1} T$.

Theorem 2.2. *Let (X, Σ, μ) be a separable nonatomic measure space. A bounded operator T on $L_p(\mu)$, $p \neq 2$, is isometrically equivalent to F on $L_p([0, 1])$ if and only if:*

(i) *T is a surjective isometry such that $T^2 = I$;*

(ii) *there exists $\sigma \in \Sigma$ such that $\Gamma(\sigma) = X \backslash \sigma$, where Γ is the σ-isomorphism induced by T.*

Proof. It is clear that if T is isometrically equivalent to F, then (i) and (ii) hold.

So conversely, assume (i) and (ii) hold. Since (X, Σ, μ) is separable and nonatomic, there exists a σ-isomorphism $\Phi: \Sigma' \to \mathcal{B}'([0, 1])$. Thus there exists a surjective isometry $J: L_p(\mu) \to L_p([0, 1])$ and $JT = RJ$ for some bounded operator R on $L_p([0, 1])$. Clearly R satisfies (i) and (ii) with respect to the σ-isomorphism it induces. Since $[0, 1]$ is a complete separable metric space, it follows that R is isometrically equivalent to F. Thus T is isometrically equivalent to F.

Remark 2.2. If $p = 2$ then conditions (i) and (ii) of Theorem 2.2 are clearly sufficient. However condition (ii) is not necessary since it is possible to find a space $L_2(X, \Sigma, \mu)$, a multiplication operator M_f on $L_2(\mu)$, defined for $g \in L_2(\mu)$ by $M_f(g) = f \cdot g$, and an isometry $U: L_2(\mu) \to L_2([0, 1])$ such that $M_f = UFU^{-1}$. The operator M_f induces the identity σ-isomorphism on Σ.

REFERENCES

1. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950. MR 11, 504.

2. J. Lamperti, *On the isometries of certain function-spaces*, Pacific J. Math. 8 (1958), 459–466. MR 21 #3764.

3. H. L. Royden, *Real analysis*, Macmillan, New York; Collier-Macmillan, London, 1963. MR 27 #1540.

4. H. A. Seid, *Cyclic multiplication operators on L_p-spaces*, Pacific J. Math. (to appear).

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