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COMPLETELY OUTER GROUPS OF AUTOMORPHISMS ACTING ON $R/J(R)$

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ABSTRACT. Let R be a ring with unit, $J(R)$ its Jacobson radical, and assume $R/J(R)$ Artinian. Let G be a finite group of automorphisms of R that induces a completely outer group on $R/J(R)$. Then R is G-Galois over the fixed ring, S, if R is projective over the usual crossed product, Δ, or, if the order of G is invertible in R, or if R is Artinian.

Let R be a ring with identity. We denote the Jacobson radical of R by $J(R)$ and we assume throughout that $R/J(R)$ is Artinian.

We assume that G is a finite group of automorphisms of R that induces a completely outer group of automorphisms on $R/J(R)$. (See Y. Miyashita [1, p. 126].) The crossed product Δ of R with G is $\Sigma\bigoplus_{\sigma\in G} Ru_\sigma$ with $(xu_\sigma)(yu_\tau) = xy^\sigma u_{\sigma\tau}$ for x and y in R.

The fixed ring S is the set of r in R such that $r^\sigma = r$ for all σ in G. We can view R as a left Δ-module by defining $xu_\sigma \cdot r = xr^\sigma$. In this way, R becomes a bi-Δ-S-module. The Jacobson radicals of Δ, R and S are denoted by $J(\Delta)$, $J(R)$ and $J(S)$ respectively.

Proposition 1. (a) $J(\Delta) = J(R)\cdot\Delta = \Delta\cdot J(R)$.
(b) $J(R)\cap S \subseteq J(S)$.

Proof of (a). Because $\sigma(J(R)) \subseteq J(R)$ for all $\sigma\in G$, $J(R)\cdot\Delta = \Delta\cdot J(R)$. Thus for any simple, left Δ-module $M \neq 0$, $J(R)M$ is a Δ-submodule of M. Now M is a finitely generated left R-module because Δ is a finitely generated R-module. Nakayama's lemma then shows $J(R)\cdot M = 0$. Since $J(R)$ annihilates every simple left Δ-module, $J(R) \subseteq J(\Delta)$.

Now G is a completely outer group of automorphisms acting on $R/J(R)$. Thus $\Delta(R/J(R), G)$ (the crossed product of $R/J(R)$ with G) has zero Jacobson radical. This was shown by T. Nakayama [4, Lemma 2, p. 204]. Now

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$\Delta(R/J(R), G) \simeq \Delta/J(R)\Delta$; hence, $J(R)\Delta \supseteq J(\Delta)$ and so $J(R)\Delta = J(\Delta)$. Similarly, $\Delta J(R) = J(\Delta)$.

Proof of (b). If x is in $J(R) \cap S$, then for some y in R, $(1-x)y = y(1-x) = 1$. So $(1-x)y^{\sigma} = y^{\sigma}(1-x) = 1$ for all σ in G. Thus $y \in S$ so $J(R) \cap S \subseteq J(S)$.

Corollary. *If a left Δ-module M is completely reducible as an R-module, then it is completely reducible as a Δ-module and conversely.*

Proof. Since $\Delta/J(\Delta)$ is a finitely generated $R/J(R)$-module, $\Delta/J(\Delta)$ is Artinian. If a left Δ-module M is completely reducible as an R-module, then $J(R)M = 0$. So $J(\Delta)M = \Delta J(R)M = 0$; then M is completely reducible as a Δ-module.

Proposition 2. *The following are equivalent.*

(1) *If $j: \Delta \to \operatorname{End} R_S$ is the map associating to each element of Δ its action on the Δ-module R (namely to $\sum x_{\sigma}u_{\sigma}$ associate the endomorphism $r \to \sum x_{\sigma}r^{\sigma}$), then j is an isomorphism.*

(2) *R is a finitely generated projective Δ-module.*

(3) *R is a Δ-generator.*

(4) *R over S is G-Galois (see [1, p. 116]).*

(5) *R^g is left Δ-isomorphic to Δ, where g is the order of G.*

Proof. (1) $\Rightarrow$ (2). Let $\overline{\Delta} = \Delta/J(\Delta)$ and $\overline{R} = R/J(R)$. We will show R_S is a generator. Now $\overline{R}$ is a finitely generated projective $\overline{\Delta}$-module, since $\overline{\Delta}$ is semisimple, Artinian. Since $\operatorname{Hom}_{\overline{\Delta}}(\overline{R}, \overline{\Delta}) \subseteq \operatorname{Hom}_{\overline{R}}(\overline{R}, \overline{\Delta}) \subseteq \overline{\Delta}$, we conclude $\operatorname{Hom}_{\overline{\Delta}}(\overline{R}, \overline{\Delta}) = \sum_{\sigma \in G} u_{\sigma}\overline{R}$. (See [2, Lemma 2.5, p. 128].) Thus there exist $f_1, \cdots, f_n \in \sum_{\sigma\in G} u_{\sigma}\overline{R}$, and $\overline{x}_1, \cdots, \overline{x}_n \in \overline{R}$ such that for all $\overline{x} \in \overline{R}$, $\sum_{i=1}^{n} f_i(\overline{x})\overline{x}_i = \overline{x}$. If $f_i(\overline{x}) = \overline{x}\sum_{\sigma}u_{\sigma}\overline{r}_i$, then $\overline{x} = \overline{x}\sum_i\sum_{\sigma}(\overline{r}_i\overline{x}_i)^{\sigma}$, for all $\overline{x} \in \overline{R}$. Thus $\overline{1} = \sum_i\sum_{\sigma}(\overline{r}_i\overline{x}_i)^{\sigma}$; let $\overline{d} = \sum_i(\overline{r}_i\overline{x}_i)$, then tr $\overline{d} = \overline{1}$. So tr $d - 1 \in J(R) \cap S \subseteq J(S)$. Thus tr $R + J(S) = S$, but $J(S)$ is small. Thus $\operatorname{tr}(R) = S$ or there is a c in R such that tr $c = 1$. We conclude that tr: $R \to S \to 0$ splits; hence R is an S-generator. Since $\operatorname{End} R_S = \Delta$, we conclude that ${}_{\Delta}R$ is finitely generated projective and $S = \operatorname{End}_{\Delta}(R)$. See K. Morita [3, Lemma 3.3, p. 100].

(2) $\Rightarrow$ (5). Now $\overline{R}^g \simeq \overline{\Delta}$ as $\overline{\Delta}$-modules, where g is the order of G (see [4, Lemma 3, p. 205]). The uniqueness of the projective cover implies R^g and Δ are Δ-isomorphic.

(5) $\Rightarrow$ (3). This is clear.

(3) $\Rightarrow$ (4). We show that R over S is G-Galois by proving R is a finitely generated, projective right S-module and $j: \Delta \to \mathrm{End}\, R_S$ is an isomorphism. But this is true by Morita's theorem. (See [3, Lemma 3.3, p. 100].)

(4) $\Rightarrow$ (1). (1) is part of the definition of G-Galois.

Proposition 3. *If R is Artinian, then R over S is G-Galois.*

Proof. By [4, Lemma 3, p. 205] there is an isomorphism $\overline{\Delta} \to \overline{R}^g$, and hence a Δ-module map $\Delta \to \overline{R}^g$. The natural epimorphism $\pi: R^g \to \overline{R}^g$ and the Δ-projectivity of Δ then gives a Δ-map $f: \Delta \to R^g$ which is an epimorphism because π is a minimal epimorphism. Since R^g and Δ are both free, rank g R-modules, they have the same R-length, and so any R-epimorphism (like f) is an isomorphism.

Proposition 4. *If a left Δ-module M is R-projective and the order of G is a unit in R, then M is Δ-projective.*

Proof. Let $\pi: N \to N'$ be a left Δ-epimorphism and $f: M \to N'$ be a left Δ-map. We want $h: M \to N$ such that $\pi h = f$. Now any Δ-module is a left R-module, so by hypothesis there exists an R-map $h': M \to N$ such that $\pi h' = f$. Let g be the order of G. Let $h(m) = \sum_{\tau\sigma=1} u_\sigma h'(u_\tau m)$. Now h is a Δ-map and $\pi h = f$.

Corollary. *If the order of G is a unit in R, then R over S is G-Galois.*

If M is a left Δ-module, define $M^G = \{m \in M | u_\sigma \cdot m = m \text{ for all } \sigma \in G\}$.

Proposition 5. *If R is Δ-projective, then $(R/J(R))^G \simeq S/J(S)$ and $J(S) = J(R) \cap S$.*

Proof. We show $\mathrm{Hom}_\Delta(R, M) \simeq M^G$ under $f \to f(1)$, if $f \in \mathrm{Hom}_\Delta(R, M)$. This is so, because $u_\sigma f(1) = f(u_\sigma \cdot 1) = f(1)$ for all $\sigma \in G$. Consider the exact sequence $0 \to J(R) \to R \to R/J(R) \to 0$. Since R is Δ-projective,

$$0 \to \mathrm{Hom}_\Delta(R, J(R)) \to \mathrm{Hom}_\Delta(R, R) \to \mathrm{Hom}_\Delta(R, R/J(R)) \to 0$$

is exact. Thus

$$0 \to S \cap J(R) \to S \to (R/J(R))^G \to 0$$

is exact. Hence $(R/J(R))^G \simeq S/S \cap J(R)$.

We now have the following situation: G is a completely outer group of automorphisms of $R/J(R)$ and the fixed ring is $S/J(R) \cap S$. We can show there is a $\overline{d} = d + J(R) \in R/J(R)$, for $d \in R$, such that $\mathrm{tr}\, d - 1 \in J(R) \cap S$. See the proof of (1) $\Rightarrow$ (2) in Proposition 2. Thus $S/J(R) \cap S$, as a left

$S/J(R) \cap S$-module, is a direct summand of $R/J(R)$, as a left $S/J(R) \cap S$-module. Since the Jacobson radical of $R/J(R)$ is zero, the Jacobson radical of $S/J(R) \cap S$ is zero. (See [1, Theorem 7.10, p. 132].) Thus $J(S) \subseteq J(R) \cap S$; hence $J(S) = J(R) \cap S$.

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