HOW SEPARABLE IS A SPACE?
THAT DEPENDS ON YOUR SET THEORY!

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ABSTRACT. A. Wilansky has raised the question of the productive behaviour of the property of having a countable set, such that each point is a sequential limit point of the set. The set-theoretic consistency and independence of the proposition that this property is preserved by products of $\aleph_1$ factors is established.

In a recent issue of the American Mathematical Monthly, A. Wilansky [13] points out that, although the product of $2^{\aleph_0}$ separable spaces is separable, it need not have a sequentially dense countable subset, even if each factor does. According to Wilansky, a set D is κ-*dense* in X, κ an infinite cardinal, if there exists a totally ordered set S with κ or fewer members, such that each $x \in X$ is the limit of a net in D defined on S. He conjectures that the product of fewer than 2^{κ} spaces, each of which has a κ-dense subset with κ points or less, also has such a subset, but that this does not hold in general for products of 2^{κ} spaces.

We shall confine ourselves here to the case $\kappa = \aleph_0$, for which we can give a complete solution to Wilansky's conjecture. We shall show that the answer depends on what axioms of set theory are assumed in addition to the "usual" ones. By the "usual" ones we mean Zermelo-Fraenkel including the Axiom of Choice, but the same arguments would work for other popular axiomatizations. The case of uncountable κ seems much more difficult — and less interesting.

Say that a space X is *sequentially separable* if there is a countable subset D of X such that each point in X is a limit of a sequence from D.

It is an easy exercise to verify that sequential separability is equivalent to $\aleph_0$-density. We shall establish the following results.

Theorem 1. *The product of countably many sequentially separable spaces is sequentially separable.*

Theorem 2. *It is consistent with the axioms of set theory that $2^{\aleph_0}$ be any regular aleph, and that the product of fewer than $2^{\aleph_0}$ sequentially separable spaces be sequentially separable.*

Theorem 2 is sharp, since a simple counting argument demonstrates that no product of continuum many sequentially separable spaces is sequentially separable if each space has at least two points and its sequential limits are unique. (Singletons and indiscrete spaces respectively prove the necessity of these conditions.) Similarly, again assuming unique sequential limits (as in Hausdorff spaces for example), if $2^{\aleph_0} < 2^{\kappa}$, no product of κ sequentially separable spaces is sequentially separable if each has at least two points. Since $2^{\aleph_0} < 2^{\aleph_1}$ is consistent with the axioms of set theory [6], so is the assumption that no product of uncountably many sequentially separable spaces with unique sequential limits and at least two points is sequentially separable.

To prove Theorems 1 and 2, we use *Martin's axiom* in its partial order form. For more on Martin's axiom, see [9], [11], [8, Chapter 5], and [12], but we recall the essentials. Let $\mathcal{P} = \langle P, \leq \rangle$ be a partial order. $p, q \in P$ are *compatible* if there is an $r \in P$, $r \leq p$, $r \leq q$. $\mathcal{P}$ satisfies the *countable chain condition* if any subset of P with no two members compatible is countable. A subset T of P is *dense* if for each $p \in P$ there is a $t \leq p$, $t \in T$. Let $\mathcal{J}$ be a collection of dense subsets of P. $G \subseteq P$ is *$\mathcal{J}$-generic* if

(1) $p \leq q$ and $p \in G$ imply $q \in G$;

(2) $p, q \in G$ imply there is an $r \in G$ with $r \leq p$, $r \leq q$;

(3) G meets each $T \in \mathcal{J}$.

Let κ be a cardinal. A_κ is the proposition that *for every countable chain condition partial order $\mathcal{P}$ and for each collection $\mathcal{J}$ of fewer than κ dense subsets of P, there is a $\mathcal{J}$-generic $G \subseteq P$.* $A_{\aleph_0}$ is true, for if $\{T_n\}_{n<\omega}$ are dense, pick $p_0 \in T_0$, $p_{n+1} \in T_{n+1}$, $p_{n+1} \leq p_n$. Then $\{p$: for some n, $p_n \leq p\}$ is $\{T_n\}_{n<\omega}$-generic. *Martin's axiom* is the assertion that A_κ *holds for each $\kappa < 2^{\aleph_0}$.* It is consistent with $2^{\aleph_0}$ being any regular uncountable aleph [11].

Theorem 3. A_κ *implies the product of no more than κ sequentially separable spaces is sequentially separable.*

Theorems 1 and 2 follow immediately. The proof of Theorem 3 is a standard forcing argument disguised, as usual, via Martin's axiom — the desired sequence is built up as the union of a generic subset of finite approximations (*forcing conditions*) to it. It is modeled on the proof in [2, p. 27] that A_κ implies that for each collection $\{B_\alpha\}_{\alpha<\kappa}$ of subsets of ω with any finite intersection of the B_α's infinite, there is an infinite $B \subseteq \omega$, such that for each α, $B - B_\alpha$ is finite. In an earlier version of this paper, we claimed that this latter proposition was sufficient to imply that the product of $\leq \kappa$ sequentially separable spaces is sequentially separable. This may well be true, but we are grateful to the referee for finding an as yet unbridged gap in that argument, leading us to substitute this one.

We leave to the reader the task of verifying that if D_i is sequentially dense in X_i, $0 \leq i \leq n < \omega$, then $\Pi_{i \leq n} D_i$ is sequentially dense in $\Pi_{i \leq n} X_i$. We thus shall assume that $\kappa < 2^{\aleph_0}$ is an infinite cardinal, and that D_α is sequentially dense in X_α, $\alpha < \kappa$. Put the discrete topology on each D_α. As the product of no more than $2^{\aleph_0}$ separable spaces, $Y = \Pi_{\alpha<\kappa} D_\alpha$ is separable [4]. Let D be countable and dense in Y. D is dense in $X = \Pi_{\alpha<\kappa} X_\alpha$; in fact we shall show it is sequentially dense, assuming A_κ.

Let π_α be the projection function from X onto X_α. Abusing notation, if f is a subset of $\omega \times X$, define

$$\pi_\alpha(f) = \{\langle n, \pi_\alpha(x)\rangle : \langle n, x \rangle \in f\}.$$

Let $x \in X$, $x = \langle x_\alpha \rangle_{\alpha<\kappa}$. There are sequences s_α from D_α, $s_\alpha \to x_\alpha$. Define a partial order $\mathcal{P} = \langle P, \leq \rangle$ as follows.

$P = \{\langle h, H \rangle: h$ is a function from a finite subset of ω into D,

$\qquad\qquad\qquad\qquad\qquad H$ is a finite subset of $\kappa\}$.

$\langle h', H' \rangle \leq \langle h, H \rangle$ if $h' \supseteq h$, $H' \supseteq H$, and for each $\alpha \in H$, $\pi_\alpha(h' - h) \subseteq s_\alpha$.

Note that any two members of P with the same first coordinate are compatible. It follows that $\mathcal{P} = \langle P, \leq \rangle$ satisfies the countable chain condition, and that the sets $E_\alpha = \{\langle h, H \rangle : \alpha \in H\}$ are dense in P.

Let $F_n = \{\langle h, H \rangle : n \in$ domain $h\}$. We shall prove that each F_n is dense. Let $\langle h, H \rangle \in P$ be given. $U_k = (\Pi\{D_\alpha: \alpha < \kappa, \alpha \notin H\}) \times (\Pi\{\{s_\alpha(k)\}: \alpha \in H\})$ is open in Y, each $k < \omega$. Suppose $n \notin$ domain h. Pick $d \in D \cap U_n$. Let $h' = h \cup \{\langle n, d \rangle\}$. Then $\langle h, H \rangle \geq \langle h', H \rangle \in F_n$.

We now are in a position to construct the desired sequence out of D converging to $x \in X$. Let G be $(\{E_\alpha\}_{\alpha<\kappa} \cup \{F_n\}_{n<\omega})$-generic. Let $g = \bigcup \{h:$ for some H, $\langle h, H \rangle \in G\}$. By compatibility, g is a function. Its do-

main is all of ω, since G meets each F_n. Thus g is a sequence from D. We will show its α-projections converge to x_α so that it converges to x. A standard argument yields that for each α, $\pi_\alpha(g) - s_\alpha$ is finite. Namely take $\langle h_0, H_0 \rangle \in G$ such that $\alpha \in H_0$. Then $\pi_\alpha(g) - s_\alpha \subseteq \pi_\alpha(h_0)$, and for given any $\langle h_1, H_1 \rangle \in G$ there is an $\langle h_2, H_2 \rangle \in G$ with $\langle h_2, H_2 \rangle \leq$ both $\langle h_0, H_0 \rangle$ and $\langle h_1, H_1 \rangle$. Then $\pi_\alpha(h_2 - h_0) \subseteq s_\alpha$, so $\pi_\alpha(h_1 - h_0) \subseteq s_\alpha$. Since $\langle h_1, H_1 \rangle$ was arbitrary, $\pi_\alpha(g - h_0) \subseteq s_\alpha$, so $\pi_\alpha(g) - s_\alpha \subseteq \pi_\alpha(h_0)$. Thus $\{n: \pi_\alpha(g)(n) \neq s_\alpha(n)\}$ is finite. Since $s_\alpha \to x_\alpha$, so does $\pi_\alpha(g)$.

The special case of whether the product of $\aleph_1$ copies of the two point discrete space is sequentially separable is particularly interesting. We have now shown this question to be independent of the axioms of set theory. It is also equivalent to a well-known problem in topology.

Theorem 4. 2^{ω_1} *is sequentially separable if and only if there is a separable normal nonmetrizable Moore space.*

The proof is obtained by putting together a number of known results.

Rothberger [10, Theorem 6] proves the equivalence of 2^{ω_1} being sequentially separable with the existence of an uncountable set X of reals such that in the subspace topology, every subset of X is an F_σ. (In Rothberger's terminology, the former proposition is stated as "there is a denumerable base for the dyadic functions on any set of power $\aleph_1$".) The latter proposition is equivalent to the existence of a separable normal nonmetrizable Moore space by Bing [1, Example E] plus Heath [7, Theorem 1]. Bukovsky [3] exhibits a model of set theory in which $2^{\aleph_0} = 2^{\aleph_1}$, but in which every separable uncountable metric space has a non-Borel subset. Thus we obtain

Corollary 5. *It is consistent with the axioms of set theory that* $2^{\aleph_0} > \aleph_1$, *but* 2^{ω_1} *is not sequentially separable.*

Foland and Kirk [5] have also worked on Wilansky's conjecture independently of the author and roughly simultaneously. Their results for uncountable cardinals can also be gotten by a straightforward extension of our methods. We need the generalized continuum hypothesis, as they do, to achieve positive results for uncountable κ, but they use it for the countable case where it is not necessary.

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