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THE HEIGHTS OF FORMAL A-MODULES

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ABSTRACT. Let A be a discrete valuation ring, finite over $\mathbf{Z}_p$, acting on a commutative formal Lie group of height h. Then h is a multiple of $|A:\mathbf{Z}_p|$; and if A acts on the tangent space by scalar multiplications, the dimension of the group is at most $h/|A:\mathbf{Z}_p|$.

Theorem 1. *Let k be a local ring of residue characteristic p. Let G be a commutative formal Lie group over k of finite height h. Let A be the ring of integers in a finite extension K of $\mathbf{Q}_p$, and suppose A acts as endomorphisms of G. Then h is divisible by $|K:\mathbf{Q}_p|$.*

Proof. Both h and the existence of A-action are unaffected by base change, so we may assume k is an algebraically closed field. Let W be the ring of Witt vectors over k; then G corresponds to a Dieudonné module M, a module over $W[F, V]$ which is free of rank h over W. Since A acts on G, it acts on M, which is thus a module over the $\mathbf{Z}_p$-tensor product $A \otimes W$. The following lemma then completes the proof.

Lemma. *M is a free module over $A \otimes W$.*

Proof. Let L be the fraction field of W. Let E be the maximal unramified subextension of K, and B its ring of integers. Let σ be the Frobenius automorphism. Then $B \otimes B$ is isomorphic to $B \times \cdots \times B$ via $b \otimes b' \mapsto \langle b\sigma^r(b')\rangle$, since E is unramified; and $\mathrm{id} \otimes \sigma$ is an automorphism of $B \otimes B$ whose powers interchange all the factors of the direct product.

Since k is algebraically closed, there is a copy of B in W, and $B \otimes W$ is similarly isomorphic to $W \times \cdots \times W$. It follows then that

$$A \otimes W \simeq A \otimes_B B \otimes W \simeq (A \otimes_B W) \times \cdots \times (A \otimes_B W),$$

and $\mathrm{id} \otimes \sigma$ still interchanges the factors. Now K is totally ramified over E, so A is generated over B by an Eisenstein element. Since L is unramified over E, the same element is still Eisenstein over W; hence $K \otimes_E L$ is a field and $A \otimes_B W$ is the valuation ring in it.

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Thus the $A \otimes W$-module M splits into the direct sum of modules over the various valuation ring factors $A \otimes_B W$; these modules are free since they are finitely generated and torsion-free. To prove M itself is free it is enough to show the summands all have the same rank. Let e be the primitive idempotent for one of the factors, so that eM is the corresponding summand. For any m in M we have $F(em) = [(\mathrm{id} \otimes \sigma)e]Fm$, so F takes eM to $[(\mathrm{id} \otimes \sigma)e]M$. Thus the powers of F map all the summands of M to each other. But $VF = p$, so F is injective on M, and hence the summands must all have the same rank.

This theorem is no surprise; it generalizes a result which was known (with a different proof) in the one-dimensional case [4, p. 470]. The Lemma, though more technical, is of more interest; for it also implies the following result, which was conjectured by Lubin.

Theorem 2. *Assume the hypotheses of Theorem* 1. *Assume also that* G *is a formal A-module in the sense of* [5], *i.e. the action which each element of* A *induces on the tangent space of* G *is a scalar multiplication. Then* $\dim G \leq h/|K : \mathbf{Q}_p|$.

Proof. Again base change allows us to assume k is an algebraically closed field, so we have a Dieudonné module M which we know satisfies the Lemma. We also know that M/VM is the tangent space of G, and its k-dimension is $\dim G$. (This assumes we are working with covariant Dieudonné modules, as in [3] or [1]. A similar argument will go through for the contravariant modules of [2].) Let $\{m_i\}$ be a basis of M as an $A \otimes W$-module. Then the images of the m_i are generators of the module M/VM. The elements of W act on this tangent space via scalar multiplications; by assumption the elements of A do so also, and thus the entire $A \otimes W$-action is via scalar multiplications. Hence the generators over $A \otimes W$ are generators over k, and the number of them is at least as large as the dimension.

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