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REGULARITY OF GENERALIZED STOCHASTIC PROCESSES AND THEIR DERIVATIVES

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ABSTRACT. If X is a generalized stochastic process which is regular in the prediction-theoretic sense then $P(d/dx)X$ is regular for a differential operator $P(d/dx)$. This is used to study sufficient conditions for regularity of stationary processes. On the other hand, an example shows that the derivative of a (nonstationary) deterministic process may be regular.

Let X be a generalized stochastic process with second moments, that is, a continuous linear map $\mathfrak{D}(\mathbf{R}^1) \rightarrow H$ where $\mathfrak{D}(\mathbf{R}^1)$ is the space of infinitely differentiable functions on $\mathbf{R}^1$ with compact support endowed with the usual inductive limit topology and H is a Hilbert space (say of square integrable random variables on a probability space). For $t \in \mathbf{R}$ let H_t = closed linear span $\{X(\phi)\colon \operatorname{supp} \phi \subset (-\infty, t)\}$; $H_{-\infty} = \bigcap_t H_t$; $H_\infty = \text{c.l.s.} \bigcup_t H_t$. If $\langle \cdot\, , \cdot \rangle$ denotes the inner product in H, then by Schwartz's kernels theorem $B(\phi \otimes \overline{\psi}) = \langle X(\phi), X(\psi) \rangle$, where for $\phi, \psi \in \mathfrak{D}(\mathbf{R}^1)$ $\phi \otimes \overline{\psi}(x, y) = \phi(x)\overline{\psi}(y)$, defines a distribution $B \in \mathfrak{D}'(\mathbf{R}^2)$. We say that X is *deterministic* if $H_{-\infty} = H_\infty$ and *regular* if $H_{-\infty} = \{0\}$. These properties depend only on the covariance distribution B: Let $N = \{\phi\colon B(\phi \otimes \overline{\phi}) = 0\}$ and let H_B be the Hilbert space obatined by completing the pre-Hilbert space $\mathfrak{D}(\mathbf{R}^1)/N$ with the inner product given by $B(\phi \otimes \overline{\psi})$. Let $X_B\colon \mathfrak{D}(\mathbf{R}^1) \rightarrow H_B$ be the natural map. Then X is regular (resp., deterministic) if and only if X_B is regular (resp., deterministic). Moreover, we see that for any distribution $B \in \mathfrak{D}'(\mathbf{R}^2)$ of covariance type, i.e., $B(\phi \otimes \overline{\phi}) \geq 0$ for all $\phi \in \mathfrak{D}(\mathbf{R}^1)$, there is g.s.p. X_B with B as its covariance distribution. If $\phi_h(x) = \phi(h + x)$ and $B(\phi_h \otimes \psi_h) = B(\phi \otimes \psi)$ for all $h \in \mathbf{R}^1$, we call X *stationary*. In this case, by the Bochner-Schwartz theorem, there exists a positive Radon measure μ on $\mathbf{R}^1$ (called the spectral measure) and an integer $N \geq 0$ such that

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(1) $B(\phi \otimes \overline{\psi}) = \int \hat{\phi}(\hat{\psi})^- d\mu, \quad \hat{\phi}(\xi) = \int e^{ix\xi}\phi(x)\,dx, \quad \int \frac{d\mu(\xi)}{(1+\xi^2)^N} < \infty.$

In case (1) holds with $N = 0$, that is, X is the g.s.p. associated with an ordinary stationary process with finite spectral measure, it is well known that if μ is absolutely continuous with respect to Lebesgue measure then X is regular if and only if

$$\int \frac{\log(d\mu/d\xi)}{1+\xi^2}\,d\xi > -\infty. \tag{2}$$

(See e.g. Deo [2].)

An analogous result for arbitrary N was stated by Rozanov [3] and Balagangadharan [1] but both of their proofs of the sufficiency of (2) contain errors. In this paper a new proof will be given based on

Proposition 1. *If X is a regular g.s.p. then $P(d/dx)X$ is regular for any differential operator $P(d/dx)$ with C^∞ coefficients.*

Proof. Suppose $P(d/dx) = \Sigma_{i=0}^n a_i(d^i/dx^i)$. Let $P^*(d/dx) = \Sigma(-1)^i a_i(d^i/dx^i)$. We define $P(d/dx)X(\phi) = X(P^*(d/dx)\phi)$. Let H_t be as above and let $H_t' = \text{c.l.s.}\{P(d/dx)X(\phi)\colon \text{supp}\,\phi \subset (-\infty, t)\} = \text{c.l.s.}\{X(P^*(d/dx)\phi)\colon \text{supp}\,\phi \subset (-\infty, t)\}$. But since $\text{supp}\,P^*(d/dx)\phi \subset \text{supp}\,\phi$, it is clear that $H_t' \subset H_t$ and hence $\bigcap H_t' \subset \bigcap H_t$.

Example. The derivative of a deterministic g.s.p. may be regular. By Deo [2] a covariance distribution B is deterministic if whenever $B(\phi \otimes \overline{\phi}) \geq |U(\phi)|^2$ for all $\phi \in \mathfrak{D}(\mathbf{R}^1)$ where $U \in \mathfrak{D}'(\mathbf{R}^1)$ vanishes on a left half-line, then $U = 0$. Let X be a g.s.p. with covariance $V \otimes \overline{V}$ where $V = 1 + F$ and F is the Heaviside function. $|V(\phi)|^2 \geq |U(\phi)|^2$ for all $\phi \in \mathfrak{D}(\mathbf{R}^1)$ and $U \neq 0$ implies $V = cU$ for some $c \neq 0$. Hence X is deterministic. Its derivative has covariance distribution $\delta_0 \otimes \delta_0$ and so is regular. The derivative of a *stationary* deterministic g.s.p. is deterministic however, since it can be shown, using Proposition 2 below, that a stationary X is deterministic if and only if

$$\int \frac{\log(d\mu_a/d\xi)}{1+\xi^2}\,d\xi = -\infty$$

where μ_a is the absolutely continuous part of μ (see Rozanov [3]). But the spectral measure of X' is $\xi^2\mu$ so the result follows from the fact that $\int((\log \xi^2)/(1+\xi^2))d\xi$ is finite.

Proposition 2. *If X is a stationary g.s.p. with absolutely continuous* *spectral measure* μ *and* (2) *holds, then* X *is regular.*

Proof. Since (1) holds for some N there is a positive $g \in L^1(\mathbf{R}^1)$ such that $d\mu/d\xi = (1+\xi^2)^N g(\xi) = (1-i\xi)^N(1+i\xi)^N g(\xi)$. Denote the inverse Fourier transform acting on tempered distributions by $\mathcal{F}^{-1}$. Then

$$\mathcal{F}^{-1}\mu = (1-(d/dx))^N(1+(d/dx))^N\mathcal{F}^{-1}g$$

and $B(\phi \otimes \overline{\psi}) = \mathcal{F}^{-1}\mu(\phi * \tilde{\psi})$ is the covariance distribution of X where $*$ denotes convolution and $\tilde{\psi}(x) = \overline{\psi(-x)}$. Let $B'(\phi \otimes \overline{\psi}) = \mathcal{F}^{-1}g(\phi * \tilde{\psi})$. Since $B'(\phi \otimes \overline{\phi}) \geq 0$, B' is the covariance distribution of some stationary g.s.p. Y with *finite* spectral measure $g(\xi)d\xi$. Then B is the covariance of $(1+(d/dx))^N Y$. Now if Y were regular, by Proposition 1, X would be also since this property depends only on the covariance. But

$$\int \frac{\log(g)\,d\xi}{1+\xi^2} = \int \frac{\log(d\mu/d\xi)}{1+\xi^2}d\xi - \int \frac{\log(1+\xi^2)^N}{1+\xi^2}d\xi > -\infty.$$

Hence Y is regular.

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