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POINTWISE IN TERMS OF WEAK CONVERGENCE[1]

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ABSTRACT. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $\mu(\Omega) < \infty$. Let X_n be a sequence of measurable functions on Ω taking values in a compact metric space M. The set of bounded stopping times τ for the X_n is a directed set under the obvious ordering. The following theorem is proved: X_n converges pointwise almost everywhere if and only if the generalized sequence $\int \phi(X_\tau)\,d\mu$ converges for every continuous function ϕ on M. The martingale theorem is proved as an application.

1. The general result. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $\mu(\Omega) < \infty$. Let M be a compact metric space, $\mathcal{C}(M)$ the space of continuous functions on M. Let $X_n = \Omega \rightarrow M$ be measurable, $n = 1, 2, \cdots$. Define $\mathcal{F}_n = \mathcal{F}(X_1, \cdots, X_n)$. A map $\tau: \Omega \rightarrow \{1, 2, \cdots\}$ will be called a stopping time if $\{\tau = n\} \in \mathcal{F}_n$ for all n. Let Γ be the collection of bounded stopping times τ.

Definition 1. A map $f: \Gamma \rightarrow \mathbf{R}$ will be called a generalized sequence. We write $f(\tau) = a_\tau$. A generalized sequence a_τ will be said to converge to a number a if for every $\epsilon > 0$ there exists $\sigma \in \Gamma$ such that $|a_\tau - a| < \epsilon$ for $\tau \geq \sigma$ (cf. [2, I.7.1]). Clearly we may choose $\sigma = n$ everywhere for some n. It is easy to see that a generalized sequence a_τ converges if and only if for every strictly increasing sequence $\tau(n) \in \Gamma$ the ordinary sequence $a_{\tau(n)}$ converges.

The basic result is Theorem 1. A similar theorem in a continuous-time setting appears in Meyer [3, Proposition 6(a), p. 232]. References to results of the same sort by F. Mertens and M. Rao are also given in [3].

Theorem 1. *The following two statements are equivalent*:

(i) X_n *converges pointwise almost everywhere on* Ω.

(ii) *For every* $\phi \in \mathcal{C}(M)$, $\int \phi(X_\tau)\,d\mu$ *is a convergent generalized sequence.*

Proof. (i) $\Rightarrow$ (ii). Fix $\phi \in \mathcal{C}(M)$. If (i) holds then $\phi(X_n)$ converges pointwise almost everywhere on Ω. The usual proof of Lebesgue's bounded convergence theorem applies without any real change.

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(ii) $\Longrightarrow$ (i). Let us first consider the case $M = [0, 1]$. Let (ii) hold and (i) not hold. We will obtain a contradiction. Let $X^- = \limsup X_n$, $X_- = \liminf X_n$. Then $\mu(\{X_- \neq X^-\}) > 0$, so there exist α and β such that $\mu(\{X_- < \alpha < \beta < X^-\}) > 0$. Choose $\epsilon > 0$ such that $\mu(\{X_- < \alpha < \beta < X^-\}) > \epsilon$. Define $\phi \in \mathcal{C}(M)$ so that $\phi = 0$ on $[0, \alpha]$, and $\phi = 1$ on $[\beta, 1]$. It is easy to see that for any n we can find bounded stopping times σ, γ such that $\gamma \geq \sigma \geq n$, $\{X_\sigma \geq \alpha\} \subseteq \{\sigma = \gamma\}$, and $\mu(\{X_\sigma < \alpha < \beta < X_\gamma\}) > \epsilon$. Then $\int\phi(X_\gamma) \geq \int\phi(X_\sigma) + \epsilon$. Hence the sequence $\int\phi(X_\tau)$ cannot converge, and a contradiction is obtained. Thus (ii) $\Longrightarrow$ (i) is proved, for the case $M = [0, 1]$. This is the standard case, since $[a, b]$ is homeomorphic to $[0, 1]$ for $-\infty \leq a < b \leq \infty$.

In the general case, if (ii) holds we use the preceding result to show that $\phi(X_n)$ converges pointwise almost everywhere on Ω, for each ϕ in $\mathcal{C}(M)$. Let $\{\phi_i\}$ be a countable dense subset of $\mathcal{C}(M)$. There exists $\Omega_1 \subseteq \Omega$ such that $\mu(\Omega - \Omega_1) = 0$ and $\phi_i(X_n(\omega))$ converges as $n \to \infty$ for each $\omega \in \Omega_1$, and all i. Hence $\phi(X_n(\omega))$ converges as $n \to \infty$ for each ω in Ω_1, and all ϕ in $\mathcal{C}(M)$. It follows that $X_n(\omega)$ converges for each ω in Ω_1. This completes the proof of Theorem 1.

Definition 2. A collection $\mathcal{H} \subseteq \mathcal{C}(M)$ will be called a separating class if for any bounded Borel measures p and q on M, $\int\phi\, dp = \int\phi\, dq$ for all $\phi \in \mathcal{H}$ implies $p = q$. A collection $\mathcal{K}$ of Borel functions on M will be said to generate a separating class if the set of all linear combinations of functions in $\mathcal{K}$ contains a separating class in $\mathcal{C}(M)$. The functions in $\mathcal{K}$ need not be continuous.

Corollary to Theorem 1. *The following condition is equivalent to* (i) *and* (ii) *of Theorem* 1:

(iii) *$\int\phi(X_\tau)d\mu$ exists and is convergent for each ϕ in a collection $\mathcal{K}$ that generates a separating class in $\mathcal{C}(M)$.*

Proof. It is enough to show (iii) $\Longrightarrow$ (ii). We may assume $\mathcal{K}$ is a separating class in $\mathcal{C}(M)$. The proof is then the same as that for ordinary sequences, applied to $X_{\tau(n)}$ for each strictly increasing sequence $\tau(n) \in \Gamma$.

Example 1. Let $M = [-\infty, \infty]$. Let $\mathcal{K}_0$ be the collection of functions ϕ, continuous on $(-\infty, \infty)$, such that:

(1) ϕ is constant on $(-\infty, a]$ and ϕ is linear and increasing on $[a, \infty)$, for some a, and

(2) ϕ is arbitrary at $\pm\infty$.

Then $\mathcal{K}_0$ generates a separating class in $\mathcal{C}(M)$, since the characteristic function of any interval $[b, \infty)$ is a bounded pointwise limit of differences of members of $\mathcal{K}_0$.

It is clearly desirable to have to check the convergence of $\int\phi(X_\tau)d\mu$ for as few ϕ's as possible. The following observation is useful in this respect, though it will not be needed in §2.

Lemma. *Let ϕ and θ be Borel functions on M such that $\int\phi(X_\tau)d\mu$ and $\int\theta(X_\tau)d\mu$ converge and such that $\int\phi \vee \theta(X_\tau)d\mu$ is bounded above. Then $\int\phi \vee \theta(X_\tau)d\mu$ converges.*

Proof. Given $\epsilon > 0$, choose N such that $\tau, \sigma \geq N$ implies $|\int\phi(X_\tau) - \int\phi(X_\sigma)| < \epsilon$ and $|\int\theta(X_\tau) - \int\theta(X_\sigma)| < \epsilon$. Choose $\tau \geq N$ such that $\int\phi \vee \theta(X_\sigma) < \int\phi \vee \theta(X_\tau) + \epsilon$ for any $\sigma \geq N$. Now suppose $\sigma \geq \tau$. Choose Borel sets A and B such that $A \cap B = \emptyset$, $A \cup B = M$, $\phi \vee \theta = \phi$ on A, and $\phi \vee \theta = \theta$ on B. Define the stopping time σ_1 by $\sigma_1 = \sigma$ on $X_\tau^{-1}(A)$ and $\sigma_1 = \tau$ on $X_\tau^{-1}(B)$. Clearly $\int\phi(X_\tau) < \int\phi(X_{\sigma_1}) + \epsilon$. Then

$$\int_{X_\tau^{-1}(A)} \phi \vee \theta(X_\tau) < \int_{X_\tau^{-1}(A)} \phi \vee \theta(X_\sigma) + \epsilon.$$

Similarly

$$\int_{X_\tau^{-1}(B)} \phi \vee \theta(X_\tau) < \int_{X_\tau^{-1}(B)} \phi \vee \theta(X_\sigma) + \epsilon.$$

Thus $\int\phi \vee \theta(X_\tau) < \int\phi \vee \theta(X_\sigma) + 2\epsilon$, so that

$$\left|\int\phi \vee \theta(X_\tau) - \int\phi \vee \theta(X_\sigma)\right| < 2\epsilon,$$

and the Lemma is proved.

2. **The martingale theorem.** Let $M = [-\infty, \infty]$. The sequence X_n of §1 will be called a submartingale if

(a) $E(|X_n|) < \infty$ for all n, and

(b) $E(X_{n+1}|X_n, \cdots, X_1) \geq X_n$ for all n.

We wish to prove the martingale theorem [1, Chapter 7]:

Theorem 2. *Let $E(|X_n|)$ be bounded. Then X_n converges pointwise almost everywhere.*

Proof. (a) and (b) imply by the submartingale stopping theorem that for $\tau, \sigma \in \Gamma$ with $\tau \geq \sigma$ we have $E(X_\tau|X_\sigma) \geq X_\sigma$. Then by Jensen's inequality we have $E(\phi(X_\tau)) \geq E(\phi(X_\sigma))$ for any $\phi \in \mathcal{K}_0$, where $\mathcal{K}_0$ is defined in Example 1. That is, $\int\phi(X_\tau)d\mu$ is a monotonic generalized sequence for each $\phi \in \mathcal{K}_0$. Since $\int|X_n|d\mu$ is bounded, so is $\int\phi(X_n)d\mu$, for each $\phi \in \mathcal{K}_0$. Hence, since $\int\phi(X_\tau)d\mu$ is monotonic, it is also bounded, for each $\phi \in \mathcal{K}_0$. As usual a bounded, monotonic generalized sequence must converge. By the corollary to Theorem 1, X_n must converge almost everywhere on Ω to a finite or infinite limit. The boundedness of $E(|X_n|)$ shows that the limit is finite almost everywhere, so Theorem 2 is proved.

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