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A FORCING PROOF OF THE KECHRIS-MOSCHOVAKIS CONSTRUCTIBILITY THEOREM

ANDREAS BLASS

ABSTRACT. We show, by forcing, that every subset of $\aleph_1$ whose codes form a Σ_2^1 set of reals must be constructible.

In [1], Kechris and Moschovakis proved the following theorem by a game-theoretic argument and expressed doubt whether it could be proved by the forcing techniques of Solovay [3].

Theorem (Kechris-Moschovakis). *Let A be a set of countable ordinals whose codes form a Σ_2^1 set of reals. Then A is constructible.*

(For details of the coding of ordinals by reals, see [1].)

The purpose of this note is to prove this theorem by forcing.

Let A be as in the hypothesis of the theorem, and let P be a Σ_2^1 formula such that, whenever a real α codes an ordinal σ,

$$(1) \qquad \sigma \in A \longleftrightarrow P(\alpha).$$

We may suppose, without loss of generality, that the statement

$$(2) \qquad \forall \alpha, \beta \; [(\alpha \text{ codes the same ordinal as } \beta) \wedge P(\beta) \longrightarrow P(\alpha)]$$

is provable in ZFC, for we may, if necessary, replace the given $P(\alpha)$ with the new Σ_2^1 formula

$$\exists \beta \; [(\alpha \text{ codes the same ordinal as } \beta) \wedge P(\beta)].$$

For each countable ordinal σ, let C_σ be the set of one-to-one finite partial functions from ω to σ. We think of C_σ as a notion of forcing (see [2]), and we write $\Vdash_\sigma$ for the associated (weak) forcing relation. The forcing language contains a name G_σ for the generic subset of C_σ and a name

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γ_σ for the well-ordering of ω (or a finite subset of ω) of length σ induced by the bijection $\bigcup G_\sigma$ from ω (or a finite subset) onto σ. Thus,

(3) $\emptyset \Vdash_\sigma G_\sigma$ is a generic (over the ground model $\check{V}$) subset of $\check{C}_\sigma$, and γ_σ is the well-ordering of $\check{\omega}$ (or a finite subset) induced by $\bigcup G_\sigma$; thus γ_σ is a code for $\check{\sigma}$.

It is easy to check that C_σ, G_σ and γ_σ are constructible functions of σ.

Consider a fixed countable ordinal σ and a code α for σ. Let C^* be a notion of forcing with respect to which every condition (weakly) forces:

(4) Every element of $\check{C}_\sigma$ belongs to some generic (over $\check{V}$) subset of $\check{C}_\sigma$.

For example, C_σ itself is such a notion of forcing, but it is perhaps easier to verify (4) if we take C^* such that the power of the continuum is collapsed to ω. With respect to any such C^*, every condition (weakly) forces the content of the following paragraph.

For every generic (over $\check{V}$) subset G of $\check{C}_\sigma$, inducing a well-ordering γ_G of $\check{\omega}$ (or a finite subset) of length $\check{\sigma}$, we have the following chain of equivalences:

$$\begin{aligned}
\check{\sigma} \in \check{A} &\longleftrightarrow \check{V} \models P(\check{\alpha}) && \text{by (1),}\\
&\longleftrightarrow P(\check{\alpha}) && \text{by Shoenfield's absoluteness theorem,}\\
&\longleftrightarrow P(\gamma_G) && \text{by (2) as both } \check{\alpha} \text{ and } \gamma_G \text{ code } \check{\sigma},\\
&\longleftrightarrow L[G] \models P(\gamma_G) && \text{by Shoenfield again.}
\end{aligned}$$

As G is generic over L and $\check{\gamma}_\sigma$ denotes γ_G in the usual interpretation of the forcing language in $L[G]$, the last formula in our chain of equivalences is implied by $L \models (\emptyset \Vdash_{\check{\sigma}} P(\check{\gamma}_\sigma))$. But conversely, if in L the empty condition does not force $P(\check{\gamma}_\sigma)$, then there is a $p \in \check{C}_\sigma$ forcing (in L) $\neg P(\check{\gamma}_\sigma)$. By (4), this p is in some generic G, so, by the chain of equivalences, $\check{\sigma} \notin \check{A}$. We have therefore

$$\check{\sigma} \in \check{A} \leftrightarrow L \models (\emptyset \Vdash_{\check{\sigma}} P(\check{\gamma}_\sigma)). \tag{5}$$

In the formula (5), which is forced with respect to C^*, all quantifiers are restricted to L. Therefore, we have (in the real world)

$$\sigma \in A \leftrightarrow L \models (\emptyset \Vdash_\sigma P(\gamma_\sigma)), \tag{6}$$

from which it immediately follows (since σ was arbitrary) that $A \in L$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48104