PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 47, Number 1, January 1975

A COMPARISON THEOREM FOR CERTAIN FUNCTIONAL DIFFERENTIAL EQUATIONS

ERNEST D. TRUE

ABSTRACT. The oscillatory character of solutions to the functional differential equation $x^{(n)}(t) + a(t)f(x(g(t))) = Q(t)$ is investigated, by comparison with the oscillatory character of solutions to $x^{(n)}(t) + s(t)f(x(t))=0$ where $s(t) \geq \gamma a(t)$, $0 < \gamma < 1$. Here, $Q(t)$ represents a bounded, oscillatory forcing function, and $g(t)$ tends to ∞ as $t \to \infty$ or $g(t) \geq t - c$ for large t but is otherwise arbitrary.

Consider the functional differential equation

$$x^{(n)}(t) + a(t)f(x(g(t))) = Q(t), \tag{1}$$

whose solutions may be compared with those of the differential equation

$$x^{(n)}(t) + s(t)f(x(t)) = 0, \tag{2}$$

where $s(t) \geq \gamma a(t)$ for all sufficiently large t, $0 < \gamma < 1$.

The questions of existence and uniqueness of solutions to initial value problems with functional arguments have been considered recently in [4], [5] and [6]. The oscillatory behaviour of solutions to functional differential equations has been examined by Waltman [8] for second order equations of the form $\ddot{x}(t) + a(t)f(x(t), x(g(t))) = 0$. Some generalizations of these results can be found in [1].

Recently, Henry [2] has submitted some numerical techniques for approximating solutions to functional differential equations. The choice of algorithm employed in approximating such solutions is often determined by prior knowledge of the behaviour of these solutions. The purpose of this work is to determine the oscillatory behaviour of solutions to (1), by comparing the solutions of (1) to the solutions of the well-known equation (2), whose solutions have been examined considerably in the literature; see for example [7].

Presented to the Society, January 26, 1973; received by the editors April 2, 1973 and, in revised form, November 23, 1973.

AMS (MOS) subject classifications (1970). Primary 34J05, 34K05; Secondary 34C10.

Key words and phrases. Functional differential equation, oscillation, comparison, forcing function.

Let F be the family of solutions of (1) which are indefinitely continuable to the right; i.e., if $x(t) \in F$, then there exists $t_0 > 0$ such that $x(t)$ exists on $[t_0, \infty)$. A solution $x(t) \in F$ is *oscillatory* if $x(t)$ has arbitrarily large zeros, and $x(t) \not\equiv 0$ on any ray $[t_1, \infty)$ for which $t_1 \geq t_0$. Also $x(t) \in F$ is *bounded* if there exists $M > 0$ such that $|x(t)| \leq M$ for all large t.

The following assumptions will be made for equation (1), the first of which is due to A. G. Kartsatos [3].

(i) $Q(t)$ is continuous on $I = [t_0, \infty)$, $t_0 > 0$, and there exists a function $R(t) \in C^n[t_0, \infty)$ for which $R^{(n)}(t) = Q(t)$ on I. Moreover, $R(t_n) = \lambda_1$, $R(t_n') = -\lambda_2$, where $-\lambda_2 \leq R(t) \leq \lambda_1$ on I, and $\{t_n\}$, $\{t_n'\}$ are any two sequences for which $\text{Lim}_{n\to\infty} t_n = \text{Lim}_{n\to\infty} t_n' = \infty$.

(ii) $f(x)$ is continuous on $(-\infty, \infty)$, nondecreasing in x and $xf(x) > 0$ for $x \neq 0$.

(iii) $g(t)$ is continuous on I and tends to ∞ as $t \to \infty$.

It is worth noting that $Q(t) \equiv 0$ on I is acceptable in (i), and $g(t)$ is not necessarily a delay or advanced argument in (iii). For example, it would be permissible to have $g(t) = t + \sin(t)$.

Theorem 1. *Assume* (i)–(iii) *hold, and in addition, suppose*

(iv) $a(t) \geq 0$ *and continuous on* I *and there exists a real number* γ, $0 < \gamma < 1$, *such that for any continuous* $s(t) \geq \gamma a(t)$, $t \geq t_0$, *the equation*

$$v^{(n)}(t) + s(t)f(v(t)) = 0 \tag{3}$$

has all its bounded solutions oscillatory.

Let $x(t) \in F$ *be bounded. Then if* n *is even,* $x(t)$ *is oscillatory, while if* n *is odd,* $x(t)$ *is oscillatory or* $\text{Lim inf}\,|x(t)| = 0$.

Proof. Suppose all bounded solutions to (3) are oscillatory, and suppose $x(t)$ is a bounded nonoscillatory solution to (1); i.e., suppose $x(t) > 0$ for all $t \geq a \geq t_0$. Since $g(t) \to \infty$, there exists a number k such that $0 < x(t) < k$, and $0 < x(g(t)) < k$ for all $t \geq \beta \geq a$ for some β.

Consider $w(t) = x(t) - R(t)$, which is a solution to

$$w^{(n)}(t) + a(t)f[w(g(t)) + R(g(t))] = 0, \qquad t \geq \beta. \tag{4}$$

Since $x(g(t)) = w(g(t)) + R(g(t)) > 0$ for $t \geq \beta$, it follows from (ii) and (1) that $f[w(g(t)) + R(g(t))] > 0$ and hence $w^{(n)}(t) < 0$. Moreover, since $x(t)$ and $R(t)$ are bounded for $t \geq \beta$, then so is $w(t)$. Thus, if n is even, then $\dot{w}(t) > 0$ for $t \geq \gamma_1 \geq \beta$ for some γ_1. Now choose m so that $t_m' \geq \gamma_1$. Then for all $t \geq t_m'$, $w(t) \geq w(t_m')$ so that

$$x(t) = w(t) + R(t) \geq w(t) - \lambda_2 \geq w(t'_m) - \lambda_2$$
$$= w(t'_m) + R(t'_m) = x(t_m) > 0.$$

Thus, $w(t) - \lambda_2 > 0$ for all large t.

If n is odd, then $\dot{w}(t) < 0$ for all $t \geq \gamma_2 \geq \beta$, and $w(t) - \lambda_2 > 0$ also; for if $w(r) - \lambda_2 \leq 0$ for some $r \geq \lambda_2$, then since $w(t)$ is decreasing, $w(t) - \lambda_2 \leq 0$ for all $t \geq r$. In particular, there is some $t'_m \geq r$ for which $w(t'_m) - \lambda_2 \leq 0$. But $w(t'_m) - \lambda_2 = w(t'_m) + R(t'_m) = x(t'_m) > 0$, which is a contradiction.

In either case, we now have

$$w(t) + R(t) \geq w(t) - \lambda_2 > 0$$

for large t.

Let $v(t) = w(t) - \lambda_2$. Then from (4), we have

$$v^{(n)}(t) + a(t)\frac{f[v(g(t)) + \lambda_2 + R(g(t))]}{f(v(t))}f(v(t)) = 0$$

or from (3) where

$$s(t) = a(t)\frac{f[v(g(t)) + \lambda_2 + R(g(t))]}{f(v(t))}.$$

We now show $s(t) \geq \gamma a(t)$ for every γ, $0 < \gamma < 1$. Since $w^{(n)}(t) < 0$ and $w(t)$ is bounded, then as before, $\dot{w}(t) > 0$ if n is even, and $\dot{w}(t) < 0$ if n is odd. Thus, $w(t)$ is monotone and therefore has a limit as $t \to \infty$.

Let $\operatorname{Lim}_{t\to\infty} w(t) = L$. Then $\operatorname{Lim}_{t\to\infty} w(g(t)) = L$.

If $L \neq \lambda_2$, then

$$\frac{f[w(g(t)) + R(g(t))]}{f(v(t))} \geq \frac{f[w(g(t)) - \lambda_2]}{f[w(t) - \lambda_2]}$$

and

$$\operatorname*{Lim}_{t\to\infty} \frac{f[w(g(t)) - \lambda_2]}{f[w(t) - \lambda_2]} = \frac{f(L - \lambda_2)}{f(L - \lambda_2)} = 1.$$

Thus, there exists a number T, such that given γ, $0 < \gamma < 1$,

$$\frac{f[w(g(t)) - \lambda_2]}{f(v(t))} > \gamma \quad \text{for all } t \geq T,$$

and hence, $s(t) \geq \gamma a(t)$. By hypothesis (iv), $v(t)$ is a solution to (3) and is, therefore, oscillatory, which contradicts $v(t) > 0$. Thus, $x(t)$ must be oscillatory.

If $L = \lambda_2$, then n is odd, for if n were even, then $\dot{w}(t) > 0$ implies $w(t)$ is increasing and $w(t) > \lambda_2$. Moreover, $w(t) = x(t) - R(t) > \lambda_2$, and for each term of the sequence $\{t'_n\}$,

$$w(t'_n) = x(t'_n) - R(t'_n) = x(t'_n) + \lambda_2 > \lambda_2.$$

Since $\operatorname{Lim}_{n\to\infty} w(t'_n) = \lambda_2$ also, then $\operatorname{Lim}_{n\to\infty} x(t'_n) = 0$ and therefore $\operatorname{Lim\,inf}\, x(t) = 0$.

The proof is similar if one assumes $x(t) < 0$.

In order to handle the unbounded solutions to (1), it becomes necessary to include some additional assumptions, and to state a lemma which is due to Grefsrud [1],

Lemma. *Assume* (i)–(iii) *are satisfied and*

(v) $g(t) \geq t - c$ *for large* t, *where* c *is any positive constant,*

(vi) *there exist positive constants* β, δ *such that* $f(\lambda x) \geq \lambda^{\beta} f(x)$ *if* $x > 0$ *and* $f(\lambda x) \leq \lambda^{\delta} f(x)$ *if* $x < 0$, λ *constant.*

Let $v(t)$ *be a nonoscillatory solution to* (3). *Then there is a real number* γ, $0 < \gamma < 1$, *such that* $f[v(g(t))]/f(v(t)) > \gamma$ *for all large* t.

Theorem 2. *If all solutions to* (3) *in Theorem* 1 *are oscillatory, then all unbounded solutions* $x(t) \in F$ *to* (1) *are also oscillatory if the additional assumptions* (v) *and* (vi) *of the lemma are included in Theorem* 1.

Proof. Suppose $x(t) \in F$ is an unbounded nonoscillatory solution to (1); i.e., suppose $x(t) > 0$ and $x(g(t)) > 0$ for all $t \geq \alpha \geq t_0$, for some α. Let $w(t) = x(t) - R(t)$, which satisfies

$$w^{(n)}(t) + a(t) f[w(g(t)) + R(g(t))] = 0, \qquad t \geq \alpha. \tag{4}$$

Since $R(t)$ is bounded, then $w(t)$ is unbounded and

$$x(t) = w(t) + R(t) \geq w(t) - \lambda_2 > 0$$

for all $t \geq \beta \geq \alpha$ for some β.

Let $v(t) = w(t) - \lambda_2$. Then $v(t)$ satisfies

$$v^{(n)}(t) + a(t) f[v(g(t)) + \lambda_2 + R(g(t))] = 0,$$

or (3) where

$$s(t) = a(t) \frac{f[v(g(t)) + \lambda_2 + R(g(t))]}{f(v(t))}, \qquad t \geq \beta.$$

We now show that $s(t) > \gamma a(t)$, $0 < \gamma < 1$. Since $\lambda_2 + R(g(t)) \geq 0$, and f is nondecreasing, then

$$\frac{f[v(g(t)) + \lambda_2 + R(g(t))]}{f(v(t))} \geq \frac{f(v(g(t)))}{f(v(t))} \geq \gamma$$

where the last inequality is a result of the lemma. Thus, $s(t) \geq \gamma a(t)$, $0 < \gamma < 1$. But $v(t)$ is then a solution to (3) and therefore, oscillatory, which contradicts $v(t) > 0$. Thus, $x(t)$ must be oscillatory. Again, the proof is similar if one assumes $x(t) < 0$ for large t.

Example 1. For n even, if $s(t) \geq \gamma/t^n$, $0 < \gamma < 1$, then all solutions to

$$x^{(n)}(t) + (1/t^n)x^3(t + \sin t) = \cos(3t - 1) \tag{5}$$

are oscillatory, since all solutions to

$$x^{(n)}(t) + s(t)x^3(t) = 0 \tag{6}$$

are oscillatory according to Theorem 1 [7].

In Theorem 2, the added restriction $g(t) \geq t - c$ has been placed on $g(t)$ in order to guarantee oscillation for the unbounded solutions. The assumptions in Theorem 1 are not sufficient to handle the unbounded solutions, as the following example, due to Waltman [8], shows:

Example 2. $x(t) = \sqrt{t}$ is a solution to

$$\ddot{x}(t) + (1/2t^2)x(t/4) = 0, \tag{7}$$

but all solutions to

$$\ddot{x}(t) + (1/2t^2)x(t) = 0 \tag{8}$$

are oscillatory. All nontrivial solutions to (8) are also unbounded, and $g(t) = t/4$ does not satisfy $g(t) \geq t - c$ in (7).

The author would like to express his gratitude for the advice and suggestions from his thesis advisor Gerald H. Ryder in the preparation of this work, and to the referee for his helpful comments.

REFERENCES

1. G. Grefsrud, *Existence and oscillation of solutions of certain functional differential equations*, Ph. D. Thesis, Montana State University, Bozeman, 1971.

2. Myron S. Henry, *Approximate solutions of functional differential equations*, Lecture Notes in Math., vol. 333, Springer-Verlag, Berlin and New York, 1972, pp. 144–152.

3. A. G. Kartsatos, *Maintenance of oscillations under the effect of a periodic forcing term*, Proc. Amer. Math. Soc. 33 (1972), 377–383.

4. R. J. Oberg, *On the local existence of solutions of certain functional-differential equations*, Proc. Amer. Math. Soc. 20 (1969), 295–302. MR 38 #2413.

5. Muril L. Robertson, *The equation* $y'(t) = F(t, y(g(t)))$, Pacific J. Math. 43 (1972), 483–491.

6. Gerald H. Ryder, *Solutions of a functional differential equation*, Amer. Math. Monthly 76 (1969), 1031–1033. MR 40 #502.

7. Gerald H. Ryder and David V. V. Wend, *Oscillation of solutions of certain ordinary differential equations of nth order*, Proc. Amer. Math. Soc. 25 (1970), 463–469. MR 41 #5710.

8. P. Waltman, *A note on an oscillation criterion for an equation with a functional argument*, Canad. Math. Bull. 11 (1968), 593–595. MR 38 #6193.

DEPARTMENT OF MATHEMATICS, MONTANA STATE UNIVERSITY, BOZEMAN, MONTANA 59715

Current address: Department of Mathematics, Norwich University, Northfield, Vermont 05663