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THE WHITEHEAD THEOREM FOR NILPOTENT SPACES

S. M. GERSTEN

ABSTRACT. An easy argument is given for the theorem of the title.

E. Dror [1] has published a far-reaching generalization of a classical theorem of J. H. C. Whitehead. An interesting case of Dror's theorem which still causes wonder among topologists is the following result.

Theorem. *If* $f: X \to Y$ *is a map of connected, pointed,* CW *complexes which induces an isomorphism on integral homology, and if* X *and* Y *are nilpotent spaces, then* f *is a homotopy equivalence.*

We remind the reader that the pointed connected space X is said to be nilpotent if (a) $\pi_1(X)$ is a nilpotent group, and (b) for each $n \geq 2$ there is a number $r_n > 0$ such that $I^{r_n}\pi_n(X) = 0$, where I is the augmentation ideal of the group ring $\mathbf{Z}[\pi_1(X)]$.

As in Dror's result, the crucial step is a reduction to a theorem of Stallings.

Lemma 1. *The map* f *of the Theorem induces an isomorphism of fundamental groups.*

Proof. Consider the spectral sequences for the fibrations

$$\begin{array}{ccccc} \tilde{X} & \longrightarrow & X & \longrightarrow & K(\pi_1(X), 1) \\ \downarrow & & \downarrow f & & \downarrow \\ \tilde{Y} & \longrightarrow & Y & \longrightarrow & K(\pi_1(Y), 1) \end{array}$$

(where $\tilde{X}$, $\tilde{Y}$ are the universal covers of X, Y). From the low-dimensional terms exact sequences, one deduces that the map $H_n(\pi_1(X, \mathbf{Z}) \to H_n(\pi_1(Y), \mathbf{Z})$ is an isomorphism for $n = 1$ and an epimorphism for $n = 2$. Since $\pi_1(X)$ and $\pi_1(Y)$ are nilpotent groups, Stallings' theorem [2] shows that $\pi_1(f): \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

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Continuing with the proof of the Theorem, we may suppose $f: X \hookrightarrow Y$ a cofibration, so $H_*(Y, X) = 0$. We prove inductively that $\pi_n(Y, X) = 0$, the induction starting with $n = 0$. Let $G = \pi_1 X = \pi_1 Y$, the latter identified via $\pi_1(f)$. Assume that $\pi_i(Y, X) = 0$ for $i < n$, $n > 0$. We write the proof in the abelian case $n > 1$ in detail (the case $n = 1$ requires only a notation change).

Lemma 2. *$\pi_n(Y, X)/G$-action $= H_n(Y, X)$.*

This is elementary, proved using the Hurewicz theorem in the universal covering spaces of X and Y.

Consider the exact sequence $\pi_n Y \xrightarrow{j} \pi_n(Y, X) \xrightarrow{\partial} \pi_{n-1}(X)$ and the short exact sequence of G-modules $0 \to \operatorname{Im} j \to \pi_n(Y, X) \to \operatorname{Im} \partial \to 0$. Since $\operatorname{Im} j$ and $\operatorname{Im} \partial$ are, respectively, quotient-modules and submodules of G-modules, by the nilpotence assumption, there is a number $m > 0$ such that

$$I^m(\operatorname{Im} j) = I^m(\operatorname{Im} \partial) = 0,$$

where I is the augmentation ideal of $\mathbf{Z}G$. Hence $I^{2m}\pi_n(Y, X) = 0$. But by Lemma 2, since $H_n(Y, X) = 0$, we have $\pi_n(Y, X) = I\pi_n(Y, X)$. Thus $\pi_n(Y, X) = 0$.

Thus the relative homotopy groups $\pi_*(Y, X)$ are all trivial, and it follows that $f: X \hookrightarrow Y$ is a homotopy equivalence.

REFERENCES

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SCHOOL OF MATHEMATICS, THE INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

Current address: Department of Mathematics, University of Illinois, Urbana, Illinois 61801