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NONEXPANSIVE MAPPINGS ON COMPACT SUBSETS OF METRIC LINEAR SPACES

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ABSTRACT. Results are presented concerning the existence of fixed points of nonexpansive selfmaps on a compact starshaped subset of a metric linear space. The results are based on a general argument of D. R. Smart and extend recent results of W. G. Dotson, Jr.

1. **Introduction.** Let (X, d) be a metric space and let S be a subset of X. A selfmap g of S is said to be *nonexpansive* if $d(g(x), g(y)) \leq d(x, y)$ for each x, y in S. If $d(g(x), g(y)) < d(x, y)$ when $x \neq y$, then g is said to be *contractive*. A result (Theorem 2) concerning the existence of fixed points for nonexpansive selfmaps of a compact starshaped subset of a metric linear space is presented in §3. This result contains as special cases recent results of W. G. Dotson, Jr. [1], [2]. The proof of Theorem 2 is based on a general lemma and a subsequent theorem (Theorem 1) presented in §2. Examples are presented which show that certain hypotheses of the Lemma cannot be relaxed.

2. The proof of the following result uses a general form of an argument due to Smart [5].

Lemma. *Let (S, d) be a compact metric space, and let g be a continuous selfmap of S. Suppose $\mathfrak{F}$ is a family of selfmaps of S satisfying:*

(i) *the identity mapping is in the uniform closure of $\mathfrak{F}$, and*

(ii) *$f \circ g$ has a fixed point in S for each f in $\mathfrak{F}$.*

Then g has a fixed point in S.

Proof. Let $\epsilon > 0$ be given. By (i), there exists f_0 in $\mathfrak{F}$ such that $d(x, f_0(x)) < \epsilon$ for each x in S. By (ii), there exists x_0 in S such that $f_0(g(x_0)) = x_0$. Then $d(x_0, g(x_0)) = d(f_0(g(x_0)), g(x_0)) < \epsilon$. Since S is compact and g is continuous, $g(u) = u$ for some u in S.

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We note that in situations where the family $\mathcal{F}$ is equicontinuous, "uniform closure" can be replaced by "pointwise closure" in (i) of the Lemma. This leads to the following generalization of Smart's result [5] from contraction mappings to contractive mappings.

Corollary. *Let (S, d) be a compact metric space. If the identity mapping is the pointwise limit of contractive selfmaps of S, then each nonexpansive selfmap of S has a fixed point.*

Proof. Let $\{f_n\}$ be a sequence of contractive selfmaps of S such that $f_n(x) \to x$ for each x in S. By the preceding remark, condition (i) of the Lemma is satisfied by the family $\mathcal{F} = \{f_n\}$. If g is a nonexpansive selfmap of S, then $f_n \circ g$ is contractive for each n. By a result of Edelstein [3], each $f_n \circ g$ has a fixed point. Hence (ii) of the Lemma is satisfied and the result follows.

We have the following generalization of a result of Dotson [2, Theorem 1].

Theorem 1. *Let (S, d) be a compact metric space, and suppose there exists a function f: $[0, 1] \times S \to S$ which satisfies*

(1) $\lim_{t \to 1} f(t, x) = x$ *for each* x *in* S,

(2) *for each* t *in* $[0, 1)$, $x \neq y$ *in* S, $d(f(t, x), f(t, y)) < d(x, y)$.

Then each nonexpansive selfmap of S has a fixed point.

Proof. Let g be a nonexpansive selfmap of S. By (1), the identity mapping is in the pointwise closure of $\mathcal{F} = \{f_t\}_{t \in [0,1]}$, where $f_t = f(t, \cdot)$. By (2), each f_t is contractive. Hence, the result follows from the previous corollary.

The hypothesis of compactness in the Lemma cannot be weakened to local compactness as the following example shows.

Example (a). Let $S = [0, 1)$ with the usual metric. Let $g: S \to S$ be defined by $g(t) = (t + 1)/2$. For each integer $n \geq 2$, let $k_n = 1 - 1/n$, and define f_n: $S \to S$ by $f_n(t) = t$ for t in $[0, k_n]$, $f_n(t) = k_n$ for t in $[k_n, 1)$. Then $(f_n \circ g)(k_n) = k_n$ for each n, and the identity on S is in the uniform closure of $\mathcal{F} = \{f_n\}_{n=2}^{\infty}$. However g has no fixed point.

Even for compact spaces, the hypothesis of uniform closure in the Lemma cannot be relaxed to pointwise closure as the following example shows.

Example (b). Let S be the countable subset of the plane defined by

$$S = \{(0, 1/n): n \geq 1\} \cup \{(0, 0)\} \cup \{(1, 1/n): n \geq 1\} \cup \{(1, 0)\}$$

with the plane metric. Let $g\colon S \to S$ be defined by

$$g((0,\ 1/n)) = (1,\ 1/n); \qquad g((1,\ 1/n)) = (0,\ 1/n);$$
$$g((0,\ 0)) = (1,\ 0); \qquad g((1,\ 0)) = (0,\ 0).$$

Let $\mathfrak{F}$ be the family of all continuous mappings f from S to S such that $f((0,\ 1/n)) = (1, 1/n)$ for some n. Then the identity mapping on S is in the pointwise closure of $\mathfrak{F}$, and $f \circ g$ has a fixed point for each f in $\mathfrak{F}$. Clearly, g has no fixed point.

3. Let $(X,\ d)$ be a metric linear space with translation invariant metric d (see [4, p. 28]). The symbol θ denotes the zero element of X. A subset S of X is *starshaped* if there exists x_0 in S such that $tx + (1-t)x_0 \in S$ whenever $t \in [0,\ 1]$, $x \in S$. We say that the metric d for X is *strictly monotone* if $x \neq \theta$ and $0 \leq t < 1$ imply $d(\theta, tx) < d(\theta, x)$.

Theorem 2. *Let $(X,\ d)$ be a metric linear space with strictly monotone metric d. Let S be a compact starshaped subset of X. Then each nonexpansive selfmap of S has a fixed point.*

Proof. Let g be a nonexpansive selfmap of S, and suppose S is starshaped at x_0. Define $f\colon [0,\ 1] \times S \to S$ by $f(t,\ x) = tx + (1-t)x_0$. Then for each x in S, $\lim_{t\to 1} f(t,\ x) = x$. If x and y are distinct members of S, and $t \in (0,\ 1)$, then

$$d(f(t,\ x),\ f(t,\ y)) = d(tx,\ ty) = d(\theta,\ t(x-y)) < d(\theta,\ x-y) = d(x,\ y).$$

It follows from Theorem 1 that g has a fixed point in S.

A p-norm $(0 \leq p \leq 1)$ on a linear space X is a nonnegative function on $X \times X$ which satisfies $\|x\| = 0$ if and only if $x = \theta$, $\|x+y\| \leq \|x\| + \|y\|$, $\|\lambda x\| = |\lambda|^p \|x\|$ for each $x,\ y$ in X and each scalar λ. Since each p-norm generates a translation invariant metric, $d(x,\ y) = \|x - y\|$, which is strictly monotone, the following is an immediate consequence of Theorem 2.

Corollary. *Let $(X,\ \|\ \|)$ be a p-normed space, and let S be a compact starshaped subset of X. Then each nonexpansive selfmap of S has a fixed point.*

For the case where the metric d is not strictly monotone, we have the following result.

Theorem 3. *Let $(X,\ d)$ be a metric linear space, and suppose that $d(\theta,\ tx) \leq d(\theta,\ x)$ whenever $|t| \leq 1$. Let S be a compact starshaped subset* *of* S. *If* $g:S \longrightarrow S$ *satisfies* $d(tg(x), tg(y)) \leq d(tx, ty)$ *for each* $x, y \in S$, $t \in [0, 1]$, *then* g *has a fixed point in* S.

Proof. The function $\rho(x, y) = \int_0^1 d(tx, ty)\,dt$ is an equivalent strictly monotone metric on S. Since g is readily verified to be nonexpansive with respect to ρ, the result follows from Theorem 2.

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