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ON THE SCALAR CURVATURE AND SECTIONAL CURVATURES OF A TOTALLY REAL SUBMANIFOLD

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ABSTRACT. For a totally real minimal submanifold of a complex space form, pinching for scalar curvature implies pinching for sectional curvatures.

Let $\tilde{M}_{n+p}(\tilde{c})$ be an $(n+p)$-dimensional complex space form with constant holomorphic sectional curvature $\tilde{c}$, complex structure $\tilde{J}$ and metric $\tilde{g}$. Let M_n be an n-dimensional real submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$ with the induced metric g. We denote by $T_x(M_n)$ and ν_x the tangent space and the normal space, respectively, of M_n at x. M_n is called the totally real submanifold of $\tilde{M}_{n+p}(\tilde{c})$ if $\tilde{J}(T_x(M_n)) \subset \nu_x$.

Let σ be the second fundamental form of the immersion, and H the length of the mean curvature vector of M_n. For a normal vector ξ on M_n, the tangential component $-A_\xi X$ of the covariant derivative $\tilde{\nabla}_X\xi$ satisfies $\tilde{g}(\sigma(X, Y), \xi) = g(A_\xi X, Y)$. We choose a local field of orthonormal frame

$$e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}, e_{1^*} = \tilde{J}e_1, \cdots, e_{n^*} = \tilde{J}e_n,$$
$$e_{(n+1)^*} = \tilde{J}e_{n+1}, \cdots, e_{(n+p)^*} = \tilde{J}e_{n+p}$$

on $\tilde{M}_{n+p}(\tilde{c})$ in such a way that, restricted to M_n, $e_1, \cdots, e_n$ are tangent to M_n. If we set $A_\alpha = Ae_\alpha$ $(\alpha, \beta = n+1, \cdots, n+p, 1^*, \cdots, (n+p)^*)$, then $\sigma(X, Y) = \sum g(A_\alpha X, Y)e_\alpha$.

Let $\tilde{R}$ and R be the curvature tensor fields of $\tilde{M}_{n+p}(\tilde{c})$ and M_n. Then

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \frac{\tilde{c}}{4}\{\tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y} + \tilde{g}(\tilde{J}\tilde{Y}, \tilde{Z})\tilde{J}\tilde{X} - \tilde{g}(\tilde{J}\tilde{X}, \tilde{Z})\tilde{J}\tilde{Y} + 2\tilde{g}(\tilde{X}, \tilde{J}\tilde{Y})\tilde{J}\tilde{Z}\}$$

and the equation of Gauss is

$$\tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) + \tilde{g}(\sigma(X, Z), \sigma(Y, W)) - \tilde{g}(\sigma(X, W), \sigma(Y, Z)),$$

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where $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ are vector fields on $\tilde{M}_{n+p}(\tilde{c})$, and X, Y, Z, W are vector fields on M_n. Since M_n is totally real in $\tilde{M}_{n+p}(\tilde{c})$ we have

$$R(X, Y; Z, W) = \frac{\tilde{c}}{4} \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}$$
$$+ \sum \{g(A_\alpha X, W)g(A_\alpha Y, Z) - g(A_\alpha X, Z)g(A_\alpha Y, W)\}.$$

The sectional curvature $K(X, Y)$ ($\{X, Y\}$ in $K(X, Y)$ is supposed to be orthonormal) and the Ricci tensor $S(X, Y)$ of M_n are then given by

$$K(X, Y) = R(X, Y; Y, X) = \frac{\tilde{c}}{4} + \sum \{g(A_\alpha X, X)g(A_\alpha Y, Y) - g(A_\alpha X, Y)^2\},$$
$$S(X, Y) = \sum_i R(X, e_i; e_i, Y)$$
$$= \frac{n-1}{4} \tilde{c} g(X, Y) + \sum (\operatorname{tr} A_\alpha) g(A_\alpha X, Y) - \sum g(A_\alpha X, A_\alpha Y).$$

Let ρ be the scalar curvature of M_n; then we have

$$\rho = \sum_i S(e_i, e_i) = \frac{n(n-1)}{4} \tilde{c} + \sum (\operatorname{tr} A_\alpha)^2 - \|\sigma\|^2 = \frac{n(n-1)}{4} \tilde{c} + n^2 H^2 - \|\sigma\|^2;$$

here $\|\sigma\|$ is the length of the second fundamental form σ, $\|\sigma\|^2 = \sum \operatorname{tr} A_\alpha^2$. If M is a minimal submanifold, then $H = 0$ and $\rho = n(n-1)\tilde{c}/4 - \|\sigma\|^2$.

We need the following algebraic lemma which is proved in [1].

Lemma. *Let $a_1, \cdots, a_n$, b be $n+1$ ($n \geq 2$) real numbers satisfying the following inequality*:

$$\left(\sum_{i=1}^n a_i\right)^2 \geq (n-1) \sum_{i=1}^n a_i^2 + b \qquad (\textit{resp.} > 1);$$

then, for any distinct i, j; $1 \leq i < j \leq n$, we have $2a_i a_j \geq b/(n-1)$ (resp.>).

We now establish the

Proposition. *Let M_n be a totally real submanifold of $\tilde{M}_{n+p}(\tilde{c})$. If the scalar curvature ρ of M_n satisfies*

$$(*) \qquad \rho \geq n(n-1)/4 \cdot \tilde{c} + n^2(n-2)/(n-1) \cdot H^2 - a$$

at a point p, then every sectional curvature of M at p is $\geq \tilde{c}/4 - a/2$.

Proof. For the frame field, $\{e_1, \cdots, e_{n+p}, e_{1*}, \cdots, e_{(n+p)*}\}$, chosen above let

$$h_{ij}^\alpha = g(A_\alpha e_i, e_j);$$

then $A_\alpha = (h_{ij}^\alpha)$, A_α is symmetric and

$$\|\sigma\|^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2, \qquad n^2H^2 = \sum_\alpha \sum_i (h_{ii}^\alpha)^2.$$

Let r be any plane section of M_n spanned by two independent tangent vectors X, Y to M_n. We choose the frame field suitably so that e_1, e_2 span r, and that e_{n+1} is parallel to the mean curvature vector of M_n. Then we have

$$K(X, Y) = K(e_1, e_2) = \frac{\tilde{c}}{4} + \sum_\alpha \{h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2\},$$

$$n^2H^2 = \left(\sum_i h_{ii}^{n+1}\right)^2.$$

The assumption (*) is equivalent to $\|\sigma\|^2 \leq n^2H^2/(n-1) + a$. Hence we have

$$\frac{1}{n-1}\left(\sum_i h_{ii}^{n+1}\right)^2 \geq \|\sigma\|^2 - a = \sum_i (h_{ii}^{n+1})^2 + \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{\alpha>n+1} (h_{ij}^\alpha)^2 - a,$$

$$\left(\sum_i h_{ii}^{n+1}\right)^2 \geq (n-1) \sum_i (h_{ii}^{n+1})^2 + (n-1)\left\{\sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{\alpha>n+1} (h_{ij}^\alpha)^2 - a\right\}.$$

By the Lemma we have

$$\begin{aligned} 2h_{11}^{n+1}h_{22}^{n+1} &\geq \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{\alpha>n+1} (h_{ij}^\alpha)^2 - a \\ &\geq 2(h_{12}^{n+1})^2 + \sum_{\alpha>n+1} \{(h_{11}^\alpha)^2 + (h_{22}^\alpha)^2 + 2(h_{12}^\alpha)^2\} - a \\ &\geq 2(h_{12}^{n+1})^2 - 2\sum_{\alpha>n+1} h_{11}^\alpha h_{22}^\alpha + 2\sum_{\alpha>n+1} (h_{12}^\alpha)^2 - a. \end{aligned}$$

Hence we obtain

$$2\sum_{\alpha\geq n+1} \{h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2\} + a \geq 0.$$

This yields

$$K(X, Y) \geq \tilde{c}/4 - a/2.$$

If M_n is minimal, then $H = 0$; the Proposition yields the

Theorem. *Let M_n be a totally real minimal submanifold of $\tilde{M}_{n+p}(\tilde{c})$. If the scalar curvature ρ of M_n satisfies $\rho \geq n(n-1)\tilde{c}/4 - a$ at a point p, then every sectional curvature of M at p is $\geq \tilde{c}/4 - a/2$.*

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