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ABSOLUTE ZERO DIVISORS AND LOCAL NILPOTENCE IN ALTERNATIVE ALGEBRAS

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ABSTRACT. It has been conjectured that absolute zero divisors generate locally nilpotent ideals in Jordan and alternative algebras. A. M. Slin'ko has recently established this result for special Jordan algebras; in this note we show how his method can be modified to establish the result for alternative algebras.

Throughout we consider alternative algebras $\mathfrak{A}$ over an arbitrary ring of scalars Φ. Recall that an element z of $\mathfrak{A}$ is an *absolute zero divisor* if $z\mathfrak{A}z = 0$. An algebra is *locally nilpotent* if every finitely-generated subalgebra is nilpotent. The *strong semiprime radical* $S(\mathfrak{A})$ is the smallest ideal $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{B}$ is *strongly semiprime* (contains no absolute zero divisors), and the *Levitzki* or *locally nilpotent radical* $L(\mathfrak{A})$ is the smallest ideal $\mathfrak{B}$ such that $\mathfrak{A}/\mathfrak{B}$ contains no locally nilpotent ideals.

Slinko's Jordan result is used to establish

Lemma. *If an alternative algebra $\mathfrak{A}$ is generated by absolute zero divisors, then the algebra $\mathfrak{L}(\mathfrak{A})$ of left multiplications of $\mathfrak{A}$ is locally nilpotent.*

Proof. The *space* $J = \{L_x \mid x \in \mathfrak{A}\}$ of left multiplications is a special Jordan subalgebra of $\mathrm{End}(\mathfrak{A})$, since by the left Moufang formula $L_xL_yL_x = L_{xyx}$ and $L_xL_x = L_{x^2}$. The *algebra* $\mathfrak{L}(\mathfrak{A})$ is the associative envelope of J in $\mathrm{End}(\mathfrak{A})$.

If z is an absolute zero divisor in $\mathfrak{A}$, then L_z is an absolute zero divisor in J: $L_zL_xL_z = L_{zxz} = 0$. Any monomial having an absolute zero divisor as a factor is itself an absolute zero divisor (using the fundamental formulas $U_{xy} = L_xU_yR_x = R_yU_xL_y$ repeatedly to break up the U-operator of the monomial until a $U_z = 0$ is reached), so if the absolute zero divisors *generate* $\mathfrak{A}$ they actually *span* $\mathfrak{A}$. Now it is not in general true that if z_i

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generate $\mathfrak{A}$ then the L_{z_i} generate J and $\mathfrak{L}(\mathfrak{A})$, but it is clear that if the z_i *span* $\mathfrak{A}$ then the L_{z_i} span J and generate $\mathfrak{L}(\mathfrak{A})$. Thus J is spanned by absolute zero divisors. By Slinko's Jordan theorem [2, pp. 713–714] the associative envelope $\mathfrak{L}(\mathfrak{A})$ of J is locally nilpotent. □

We are now ready to state the alternative version of Slinko's theorem.

Slinko's theorem I. *An alternative algebra which is generated by absolute zero divisors is locally nilpotent.*

Proof. Let $\mathfrak{B}$ be a finitely-generated subalgebra of an algebra $\mathfrak{A}$ generated by absolute zero divisors z_i; we must prove $\mathfrak{B}$ is nilpotent. Now each of the finitely many generators of $\mathfrak{B}$ is by hypothesis a polynomial in a finite number of the absolute zero divisors z_i, so $\mathfrak{B}$ is contained in the subalgebra $\mathfrak{C}$ generated by all these finitely many z_i. It suffices to prove $\mathfrak{C}$ is nilpotent. Thus the theorem is equivalent to

Slinko's theorem II. *An alternative algebra generated by a finite number of absolute zero divisors is nilpotent.*

Proof. Suppose $\mathfrak{A}$ is generated by absolute zero divisors $z_1, \cdots, z_n$. We know [1, Proposition 3, p. 290] that every element of $\mathfrak{A}^k$ is a linear combination of "2nd order monomials" of the form $w_1(w_2(\cdots w_r))$, where each $w_i = z_{i_1}(z_{i_2}(\cdots z_{i_s}))$ is a "1st order monomial" in the generators z_j, and the degrees of the w_i add up to at least k: $\partial w_1 + \partial w_2 + \cdots + \partial w_r \geq k$. By the lemma there is an $N = N(z_1, \cdots, z_n)$ such that

$$L_{z_{i_1}} L_{z_{i_2}} \cdots L_{z_{i_{s-1}}} = 0 \text{ for } s \geq N,$$

so

$$w_i = L_{z_{i_1}} L_{z_{i_2}} \cdots L_{z_{i_{s-1}}} (z_{i_s}) = 0$$

if $\partial w_i = s \geq N$. Consider the finite number $w_1, \cdots, w_m$ of 1st order monomials w_i of degree $< N$. Once more by the lemma there is an $M = M(w_1, \cdots, w_m)$ such that

$$w_{i_1}(w_{i_2}(\cdots w_{i_r})) = L_{w_{i_1}} L_{w_{i_2}} \cdots L_{w_{i_{r-1}}} (w_{i_r}) = 0 \text{ for } r \geq M.$$

But then $\mathfrak{A}^{NM}$ is spanned by $w_1(w_2(\cdots w_r))$ for $\partial w_1 + \cdots + \partial w_r \geq NM$, where this monomial vanishes unless all $\partial w_i < N$ ($w_i = 0$ if $\partial w_i \geq N$), and if all $\partial w_i < N$ then $rN > \partial w_1 + \cdots + \partial w_r \geq NM$ implies $r \geq M$, so $w_1(w_2(\cdots w_r)) = 0$ anyway. Thus $\mathfrak{A}^{NM} = 0$ and $\mathfrak{A}$ is nilpotent. □□

From this we can immediately derive some corollaries.

Corollary. *The ideal* $Z(\mathfrak{A})$ *generated (or spanned) by the absolute zero divisors is contained in the Levitzki radical,* $Z(\mathfrak{A}) \subset L(\mathfrak{A})$. □

Corollary. *If* $\mathfrak{A}$ *contains no locally nilpotent ideals, it is strongly semiprime.* □

Corollary. $S(\mathfrak{A}) \subset L(\mathfrak{A})$. □

Thus in an arbitrary alternative algebra we have the chain of radicals

$$P(\mathfrak{A}) \subset S(\mathfrak{A}) \subset L(\mathfrak{A}) \subset N(\mathfrak{A}) \subset J(\mathfrak{A})$$

for P the prime radical, N the nil radical, and J the Jacobson-Smiley radical.

Since simple algebras are not locally nilpotent they have no locally nilpotent ideals, hence

Corollary. *A simple alternative algebra contains no absolute zero divisors.* □

Of course, this also follows from inspecting the known classification of simple alternative algebras.

Another result which follows without appeal to the classification of simple algebras is the

Corollary. *A simple, commutative, alternative algebra is a field.*

Proof. It suffices to prove associativity. A simple commutative alternative algebra cannot contain nilpotent elements, for if it contained nilpotent elements it would contain elements with $z^2 = 0$, and by commutativity such z would be absolute zero divisors ($zxz = zzx = z^2x = 0$ for all x), contrary to our previous corollary. But any associator is nilpotent, $[x, y, z]^2 = 0$, so all associators must vanish and the algebra is associative. □

Yet another result whose proof can be simplified is one due to Zhevlakov, Ng, and others:

Corollary. *If* $\mathfrak{A}$ *is a simple alternative algebra, then* $\mathfrak{A}^+$ *is simple as a quadratic Jordan algebra.*

Proof. Suppose $\mathfrak{B}$ were a proper Jordan ideal of $\mathfrak{A}^+$. Then its kernel (the largest alternative ideal contained in $\mathfrak{B}$) must be zero by simplicity of $\mathfrak{A}$. Now it is easy to verify that in general

$$\text{Ker } \mathfrak{B} = \{b \in \mathfrak{B} | b\mathfrak{A} \subset \mathfrak{B}\} = \{b \in \mathfrak{B} | \mathfrak{A}b \subset \mathfrak{B}\}$$

for a Jordan ideal $\mathfrak{B}$, hence

$$(bxb)a = b\{x(ba)\} = U_{b,ba}x - (ba)(xb) = U_{b,ba}x - U_b(ax) \in U_{\mathfrak{B},\mathfrak{A}}\mathfrak{A} - U_{\mathfrak{B}}\mathfrak{A} \subset \mathfrak{B}$$

shows $U_{\mathfrak{B}}\mathfrak{A} \subset \text{Ker } \mathfrak{B}$. Thus when the kernel is zero, all elements of $\mathfrak{B}$ are absolute zero divisors, and therefore by the corollary, $\mathfrak{B} = 0$. □

Remark. The result we really want, of course, is $P(\mathfrak{A}) = S(\mathfrak{A})$ rather than just $S(\mathfrak{A}) \subset L(\mathfrak{A})$; we want to know that a semiprime alternative algebra contains no absolute zero divisors so that strong semiprimeness is equivalent to semiprimeness. Kleinfeld [3] has proved this for characteristic $\neq 3$ situations (where $\mathfrak{A}$ has no 3-torsion or $3\mathfrak{A} = \mathfrak{A}$), but no one has been able to extend it to the general case. This is the main stumbling block to a classification of prime and semiprime alternative algebras.

For many applications (such as the above corollaries) the weaker inclusion $S(\mathfrak{A}) \subset L(\mathfrak{A})$ suffices. In characteristic $\neq 3$ the corollaries also follow from Kleinfeld's result. □

Further remark. Professor Michael Slater has pointed out that Slinko's theorem can also be derived quickly from the known structure theory of alternative rings. The advantage of the present direct proof is that it bypasses the difficult part of the structure theory (due to Shirshov) concerning p.i. rings; indeed, it can be used in place of Shirshov's work to derive that structure theory (as will appear in a forthcoming paper of Professor Slater). It seems that the really essential part of the Shirshov machinery is the part used in proving Slinko's theorem (see the lemma below). □

Slinko's proof was given for linear Jordan algebras over a field of characteristic $\neq 2$. His proof can be modified to work for quadratic Jordan algebras over an arbitrary ring of scalars. To make this paper self-contained we will repeat (and thereby translate from Russian) the part of Slinko's proof applying to alternative algebras, making the minor modifications necessary for the case of an arbitrary ring of scalars.

Lemma. *Let $z_1, \cdots, z_n$ be absolute zero divisors in an alternative algebra. Then there is an integer $N = 2^n(n+1)!$ such that any product $L_{z_{i_1}} \cdots L_{z_{i_N}}$ of left multiplications L_{z_i} of length N vanishes.*

Proof. We induct on n, $n = 0$ being vacuous. Assume the result for $n - 1$ absolute zero divisors, so any monomial in $L_{z_1}, \cdots, L_{z_{n-1}}$ of length

$N_0 = 2^{n-1}n!$ vanishes. We claim that any monomial in $L_{z_1}, \cdots, L_{z_n}$ of length $N = 2^n(n+1)!$ vanishes.

We can write any such monomial as

$$w(L_{z_1}, \cdots, L_{z_n}) = L_{z_{i_1}} \cdots L_{z_{i_N}}$$

where $w(x_1, \cdots, x_n) = x_{i_1} \cdots x_{i_N}$ is a word on the alphabet of letters $x_1, \cdots, x_n$. We order this alphabet in the natural way, $x_1 < x_2 < \cdots < x_n$, and induct on the lexicographic order of w. This induction gets off the ground since the lexicographically lowest word of length N is x_1^N, and $L_{z_1}^N = L_{z_1^N} = 0$ by $z_1^e = z_1(z_1^{e-2})z_1 = 0$ if $e \geq 3$ and z_1 is an absolute zero divisor.

Assume the result for lexicographically lower words w'. Write

$$w = w_0 x_n^{e_1} w_1 \cdots w_{r-1} x_n^{e_r} w_r$$

for exponents $e_i \geq 1$ and words $w_i(x_1, \cdots, x_{n-1})$ not involving x_n where $w_1, \cdots, w_{r-1}$ are nonempty (we allow $w_0 = 1$ or $w_r = 1$). The word w will vanish when evaluated at the L_{z_i}, $w(L_{z_1}, \cdots, L_{z_n}) = 0$, if any exponent $e_i \geq 3$ (since $L_{z_n}^e = L_{z_n^e} = 0$ if $e \geq 3$) or if any word w_i has length $\geq N_0$ (by the induction hypothesis for $n-1$ absolute zero divisors). Thus we may assume $e_i \leq 2$ and $\partial w_i < N_0$. Then

$$(n+1)2^n n! = N = \partial w = \sum_1^r e_i + \sum_0^r \partial w_i \leq \sum_1^r 2 + \sum_0^r N_0$$

$$= 2r + (r+1)N_0 < (r+1)(2 + N_0) \leq (r+1)2N_0 = (r+1)2^n n!$$

forces $n < r$.

If one of the w_i for $1 \leq i \leq r-1$ has degree 1, say $w_i = x_j$, then again $w(L_{z_1}, \cdots, L_{z_n}) = 0$ since already

$$L_{z_n}^{e_i} L_{z_j} L_{z_n}^{e_{i+1}} = L_{z_n}^{e_i - 1}(L_{z_n} L_{z_j} L_{z_n}) L_{z_n}^{e_{i+1}-1}$$

$$= L_{z_n}^{e_i - 1}(L^{\bullet}_{z_n z_j z_n}) L_{z_n}^{e_{i+1}-1} = 0$$

if z_n is an absolute zero divisor. Thus we may assume the monomials $w_1, \cdots, w_{r-1}$ have degree ≥ 2; there are $r - 1 \geq n$ of these monomials and only $n-1$ variables $x_1, \cdots, x_{n-1}$ appearing in them, so two of them must end in the same variable x_k: $w_i = w_i' x_k$, $w_j = w_j' x_k$ for $1 \leq i < j \leq r-1$. We will rearrange the letters between these two x_k, obtaining lexicographically lower words.

By [1, p. 286] there is a Jordan monomial $p(x_1, \cdots, x_n)$ having as lexicographically leading monomial

$$v = x_n^{e_{i+1}} w_{i+1} \cdots x_n^{e_j} w'_j$$

(here it is crucial that v begins with an x_n and ends with a lower letter, since $w_j = w'_j x_k$ for degree ≥ 2 implies w'_j is of degree ≥ 1 on the letters $x_1, \cdots, x_{n-1}$); we write

$$p(x_1, \cdots, x_n) = v + \sum v_\alpha$$

for associative words v_α lexicographically lower than v. Then $w = w'vw''$ for

$$w' = w_0 w_n^{e_1} \cdots x_n^{e_i} w_i = (w_0 \cdots w'_i) x_k = u' x_k$$

and

$$w'' = x_k x_n^{e_{j+1}} w_{j+1} \cdots x_n^{e_r} w_r = x_k u'',$$

where

$$\begin{aligned} w' p(L_{z_1}, \cdots, L_{z_n}) w'' &= u' L_{z_k} p(L_{z_1}, \cdots, L_{z_n}) L_{z_k} u'' \\ &= u' L_{z_k} L_{p(z_1, \dots, z_n)} L_{z_k} u'' \quad \text{(by [1, Proposition 2, p. 285])} \\ &= u' L_{z_k p(z_1, \dots, z_n) z_k} u'' = 0 \end{aligned}$$

if z_k is an absolute zero divisor. This allows us to replace v by the lower v_α in w,

$$\begin{aligned} w(L_{z_1}, \cdots, L_{z_n}) &= w' v(L_{z_1}, \cdots, L_{z_n}) w'' \\ &= w' p(L_{z_1}, \cdots, L_{z_n}) w'' - \sum w' v_\alpha (L_{z_1}, \cdots, L_{z_n}) w'' \\ &= - \sum w' v_\alpha (L_{z_1}, \cdots, L_{z_n}) w'' \\ &= - \sum w_\alpha (L_{z_1}, \cdots, L_{z_n}) \end{aligned}$$

for $w_\alpha = w' v_\alpha w''$ lexicographically lower than $w = w' v w''$ (but still of the same length N). By lexicographic induction these $w_\alpha(L_{z_1}, \cdots, L_{z_n}) = 0$, hence $w(L_{z_1}, \cdots, L_{z_n}) = 0$ too. This completes both inductions, so any $L_{z_{i_1}} \cdots L_{z_{i_N}} = 0$. □

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