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TRIPLES ON REFLECTIVE SUBCATEGORIES OF FUNCTOR CATEGORIES

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ABSTRACT. We show that if $\mathbf{S}$ is a cocontinuous triple on a full reflective subcategory of a functor category then the category of $\mathbf{S}$-algebras is again a full reflective subcategory of a functor category.

This note should be considered an addendum to [4], and definitions for all of the terminology and concepts we use can be found there.

We shall also fix $\mathbf{V}$ to be a closed bicomplete category and, in this note, all of the category theory is done relative to $\mathbf{V}$.

In [4], we have shown the following:

(I) If $\mathbf{C}$ is a small category and $\mathbf{T}$ is a cocontinuous triple on the functor category $\mathbf{V}^{\mathbf{C}}$, then there is a small category $\mathbf{C}'$ and a functor f: $\mathbf{C} \to \mathbf{C}'$ so that

(a) $\mathbf{T}$ is the triple induced by the adjoint pair (f^*, f^l): $\mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$, where f^*: $\mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$ is the functor induced by f and f^l is the left adjoint of f^*;

(b) the adjoint pair (f^*, f^l) is tripleable, so that there is an equivalence of categories $\mathbf{V}^{\mathbf{C}'} \cong (\mathbf{V}^{\mathbf{C}})^{\mathbf{T}}$, where $(\mathbf{V}^{\mathbf{C}})^{\mathbf{T}}$ is the category of $\mathbf{T}$-algebras.

(II) If $\mathbf{C}$ is a small category and $\mathbf{T}$ is any triple on $\mathbf{V}^{\mathbf{C}}$, there is a unique cocontinuous triple $\hat{\mathbf{T}}$ on $\mathbf{V}^{\mathbf{C}}$ and a map of triples τ: $\hat{\mathbf{T}} \to \mathbf{T}$ so that, if R: $\mathbf{C}^{\circ} \to \mathbf{V}^{\mathbf{C}}$ denotes the right Yoneda embedding of $\mathbf{C}^{\circ}$ into the representable functors of $\mathbf{V}^{\mathbf{C}}$, τR is the identity (we shall refer to $\hat{\mathbf{T}}$ as the *cocontinuous approximation* to $\mathbf{T}$).

In this paper, we shall prove the following

1. **Theorem.** *Suppose* $\mathbf{C}$ *is a small category,* $\mathbf{A}$ *is a full reflective subcategory of* $\mathbf{V}^{\mathbf{C}}$ *(i.e. the inclusion functor of* $\mathbf{A}$ *to* $\mathbf{V}^{\mathbf{C}}$ *has a left adjoint) and* $\mathbf{S}$ *is a cocontinuous triple on* $\mathbf{A}$. *Then there is a small catego-*

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ry $\mathbf{C}'$ *for which the category of* $\mathbf{S}$*-algebras* $\mathbf{A}^{\mathbf{S}}$ *is a full reflective subcategory of* $\mathbf{V}^{\mathbf{C}'}$.

The basic idea of the proof of this theorem is as follows: as we shall see, $\mathbf{S}$ induces a triple $\mathbf{T}$ on $\mathbf{V}^{\mathbf{C}}$ in an obvious way; we use (II) to construct the cocontinuous approximation $\hat{\mathbf{T}}$ of $\mathbf{T}$, and then we apply (I) to $\hat{\mathbf{T}}$ to obtain the desired category $\mathbf{C}'$.

The rest of this paper is devoted to showing that the above outline does indeed give a proof for the theorem.

Proof of the theorem. Let $\mathbf{A}$ and $\mathbf{B}$ be categories and suppose (i, r): $\mathbf{A} \to \mathbf{B}$ is an adjoint pair from $\mathbf{A}$ to $\mathbf{B}$ with unit u: $1_{\mathbf{B}} \to ir$ and counit e: $ri \to 1_{\mathbf{A}}$. We shall let $\mathbf{R} = (R, u, m)$ denote the triple on $\mathbf{B}$ induced by (i, r) (so that $m = ier$).

Suppose $\mathbf{S} = (S, \eta', \mu')$ is a triple on $\mathbf{A}$. Then $\mathbf{S}$, together with (i, r), induces a triple $\mathbf{T} = (T, \eta, \mu)$ on $\mathbf{B}$ by letting $T = iSr$, $\eta = (i\eta' r) \cdot u$ and $\mu = (i\mu' r) \cdot (iSeSr)$. Equivalently, $\mathbf{T}$ is the triple induced by the adjoint pair obtained by composing the adjoint pair (i, r): $\mathbf{A} \to \mathbf{B}$ with the adjoint pair $(U^{\mathbf{S}}, F^{\mathbf{S}})$: $\mathbf{A}^{\mathbf{S}} \to \mathbf{A}$, where $U^{\mathbf{S}}: \mathbf{A}^{\mathbf{S}} \to \mathbf{A}$ is the usual "underlying" functor from the category of $\mathbf{S}$-algebras to $\mathbf{A}$ and $F^{\mathbf{S}}$ is the usual "free" functor.

One obtains easily the following facts:

(1) there is a comparison functor $\tilde{i}: \mathbf{A}^{\mathbf{S}} \to \mathbf{B}^{\mathbf{T}}$;

(2) there is a map of triples θ: $\mathbf{R} \to \mathbf{T}$ given by $\theta = i\eta' r$.

2. **Proposition.** *Suppose* $\mathbf{A}$ *is a full reflective subcategory of* $\mathbf{B}$, *i.e., the inclusion functor* i: $\mathbf{A} \to \mathbf{B}$ *has a left adjoint* r, $\mathbf{S} = (S, \eta', \mu')$ *is a triple on* $\mathbf{A}$, *and* $\mathbf{T} = (T, \eta, \mu)$ *is the triple on* $\mathbf{B}$ *induced by* $\mathbf{S}$ *and* (i, r). *Then*

(a) *the comparison functor* $\tilde{i}: \mathbf{A}^{\mathbf{S}} \to \mathbf{B}^{\mathbf{T}}$ *of* (1) *is an equivalence of categories, and*

(b) *if* $\mathbf{R} = (R, u, m)$ *is the idempotent triple on* $\mathbf{B}$ *induced by the adjoint pair* (i, r), *then* $T = TR = RT$ *and the map of triples* θ: $R \to T = RT$ *of* (2) *is given by* $\theta = R\eta$. *Furthermore,* $\mu \cdot \theta T = \mu \cdot T\theta = 1_T$.

Proof. (a) Follows from Beck's tripleability theorem (see [7]). For (b), we have $RT = iriSr = iSr = T$, as the counit e: $ri \to 1_{\mathbf{A}}$ is the identity. Similarly $TR = T$. $R\eta = ir(i\eta' r \cdot u) = iri\eta' r \cdot iru = i\eta' r$ (as $ri = 1_{\mathbf{A}}$ and $ru = 1_r) = \theta$. $\mu \cdot \theta T = i\mu' r \cdot iSeSr \cdot i\eta' riSr = i\mu' r \cdot i\eta' Sr$ (as $e = 1$ and $ri = 1_{\mathbf{A}}) = i(\mu' \cdot \eta' S)r = i 1_S r$(as $\mathbf{S}$ is a triple) $= 1_T$. Similarly $\mu \cdot T\theta = 1_T$. □

For the rest of this paper, let us make the following hypotheses.

(i) $\mathbf{B}$ is cocomplete and there is a small category $\mathbf{C}$ and a functor k: $\mathbf{C} \to \mathbf{B}$ which is dense in $\mathbf{B}$ (see [7]);

(*) (ii) there is an adjoint pair (i, r): $\mathbf{A} \to \mathbf{B}$ whose counit is the identity (so that $\mathbf{A}$ is equivalent to a full reflective subcategory of $\mathbf{B}$) and $\mathbf{R} = (R, u, m)$ is the idempotent triple on $\mathbf{B}$ induced by (i, r);

(iii) $\mathbf{S} = (S, \eta', \mu')$ is a cocontinuous triple on $\mathbf{A}$ and, for $\mathbf{T} = (T, \eta, \mu)$, the triple on $\mathbf{B}$ induced by $\mathbf{S}$ and (i, r), there is a cocontinuous triple $\hat{\mathbf{T}} = (\hat{T}, \hat{\eta}, \hat{\mu})$ on $\mathbf{B}$ and a map of triples τ: $\hat{\mathbf{T}} \to \mathbf{T}$ with $\tau k = 1$.

We note that if $\mathbf{C}$ is a small category, $\mathbf{A}$ is a full reflective subcategory of $\mathbf{V}^{\mathbf{C}}$, and $\mathbf{S}$ is a cocontinuous triple on $\mathbf{A}$ (as in the hypotheses of 1), then $\mathbf{B} = \mathbf{V}^{\mathbf{C}}$, k the right Yoneda embedding R: $C^{\circ} \to \mathbf{V}^{\mathbf{C}}$, and $\hat{\mathbf{T}}$ the cocontinuous approximation of $\mathbf{T}$ satisfy the above hypotheses $(*)$.

Recall that for $\mathbf{X}$ a category and for $\mathbf{R} = (R, u, m)$ and $\mathbf{T} = (T, \eta, \mu)$ two triples on $\mathbf{X}$, the *composite triple of* $\mathbf{R}$ *and* $\mathbf{T}$ is a triple $\mathbf{RT} = (RT, u\eta, v)$ for which $R\eta$: $R \to RT$ and uT: $T \to RT$ are maps of triples and for which $v \cdot (R\eta uT) = 1_{RT}$.

3. **Proposition.** *Under the hypotheses* $(*)$, $\mathbf{T}$ *is a composite triple of* $\mathbf{R}$ *and* $\hat{\mathbf{T}}$.

Proof. r, being an adjoint, is cocontinuous. Since $rT = riSr = Sr$ and since $r\tau$: $r\hat{T} \to rT = Sr$ is a natural transformation between cocontinuous functors for which $r\tau k = 1$, and since k is assumed dense, it follows that $r\tau$ is an isomorphism of functors. Hence $R\tau$: $R\hat{T} \to RT = T$ is an isomorphism of functors. Let $\mathbf{RT}$ be $R\hat{T}$ with the triple structure induced by that of T *via* the isomorphism $R\tau$. Since $\hat{\eta} \cdot \tau = \eta$, one has $(R\tau) \cdot (u\hat{\eta}) = u\eta = \eta$ so that $u\hat{\eta}$ is the unit of $\mathbf{R\hat{T}}$.

$R\hat{\eta}$: $R \to R\hat{T}$ is a map of triples, since $R\tau \cdot R\hat{\eta} = R(\tau \cdot \hat{\eta}) = R\eta = \theta$ is a map of triples.

We now show that $u\hat{T}$: $\hat{T} \to R\hat{T}$ is a map of triples. Now $(R\tau \cdot \hat{T})k = R\tau k \cdot u\hat{T}k = 1 \cdot uTk = uiSrk = 1$ (as $ui = 1$ and $rk = 1$). Since $\hat{T}$ is cocontinuous and k is dense, $\hat{T}$ is the left Kan extension of $\hat{T}k$ along k (see [7, p. 232]). The universal property of left Kan extensions gives us that $R\tau \cdot n\hat{T} = \tau$, and since τ is a map of triples, so is $u\hat{T}$.

Finally, if v is the multiplication of $\mathbf{R\hat{T}}$ (induced by μ), we have $[v \cdot (R\hat{\eta}n\hat{T})] = 1_{R\hat{T}}$ since, by §2, $\mu \cdot \theta T = 1$ so that the diagram

$$\begin{tikzcd}
R\hat{T} \arrow[r, "\cong", "R\tau"'] \arrow[d, "R\hat{\eta}n\hat{T}"'] \arrow[dr, "\theta\tau"] & RT \arrow[d, "\theta T"] \arrow[dr, equal] & \\
R\hat{T}R\hat{T} \arrow[r, "\cong", "R\tau R\tau"'] & TT \arrow[r, "\mu"'] & T
\end{tikzcd}$$

commutes. □

4. **Corollary.** *Under the hypotheses* (∗), $\mathbf{A}^{\mathbf{S}}$ *is equivalent to a full reflective subcategory of* $\mathbf{B}^{\hat{\mathbf{T}}}$.

Proof. From [2, p. 122] and §3 we have a lifting of R to a triple $\tilde{R}$ on $\mathbf{B}^{\hat{\mathbf{T}}}$ and an isomorphism of categories $\Phi: (\mathbf{B}^{\hat{\mathbf{T}}})^{\tilde{\mathbf{R}}} \xrightarrow{\sim} \mathbf{B}^{\mathbf{R}\hat{\mathbf{T}}}$. But $\mathbf{R}\hat{\mathbf{T}} \cong \mathbf{T}$ by §3 so that $\mathbf{B}^{\mathbf{R}\hat{\mathbf{T}}} \cong \mathbf{B}^{\mathbf{T}} \cong \mathbf{A}^{\mathbf{S}}$ by §2. Since the underlying functor from $\mathbf{B}^{\hat{\mathbf{T}}}$ to $\mathbf{B}$ is faithful and $\mathbf{R}$ is idempotent, the lifting $\tilde{\mathbf{R}}$ is idempotent. □

We note that Theorem 1 now follows from this corollary.

A problem arising from this theorem is the following: if $\mathbf{C}$ is a small category, $\mathbf{A}$ is a full reflective subcategory of $\mathbf{V}^{\mathbf{C}}$, and $\mathbf{S}$ a cocontinuous triple on $\mathbf{A}$, then is $\mathbf{A}^{\mathbf{S}}$ a full reflective subcategory of $\mathbf{V}^{\mathbf{C}'}$ "of the same type"? For example, if $\mathbf{V} = \mathbf{Ab}$ (the category of abelian groups), $\mathbf{C}$ a small abelian category, $\mathfrak{L}$ the category of left exact functors from $\mathbf{C}$ to $\mathbf{Ab}$, and $\mathbf{S}$ a cocontinuous triple on $\mathfrak{L}$, then $\mathfrak{L}^{\mathbf{S}}$ is a full reflective subcategory of $\mathbf{Ab}^{\mathbf{C}'}$ for some preadditive category $\mathbf{C}'$ by our theorem, but is $\mathbf{C}'$ abelian and is $\mathfrak{L}^{\mathbf{S}}$ the category of left exact functors from $\mathbf{C}'$ to $\mathbf{Ab}$?

The following is an example where the answer to this question is positive.

Let $\mathbf{C}$ be a small category and let J be a topology on $\mathbf{C}$ making $(\mathbf{C}, J)$ into a site (as in [5, Definition 1.2, pp. 256–303]). Let $\mathbf{A}$ be the category of sheaves of sets on $\mathbf{C}$, so that $\mathbf{A}$ is a full reflective subcategory of the functor category (i, r): $\mathbf{A} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, where i: $\mathbf{A} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is the inclusion functor, then $\mathbf{R}$ is a left exact idempotent triple (where "left exact" means that the functor of $\mathbf{R}$ preserves finite limits).

Now $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is an example of an elementary topos (as in [6, p. 5]) and one sees that the topologies J on $\mathbf{C}$ are in one-to-one correspondence with the topologies on the elementary topos $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ (as defined in [6]). One can then show (using [6, Proposition 3.22, p. 70]) that the assignment

$J \mapsto \mathbf{R}$ as in the previous paragraph gives a one-to-one correspondence between topologies J on $\mathbf{C}$ and left exact idempotent triples $\mathbf{R}$ on $\mathrm{Sets}^{\mathbf{C}^{\mathrm{op}}}$.

5. **Theorem.** *Let* $\mathbf{C}$ *be a small category,* J *a topology on* $\mathbf{C}$, $\mathbf{A}$ *the category of sheaves of sets on* $\mathbf{C}$ *with respect to* J, *and* $\mathbf{S}$ *a cocontinuous triple on* $\mathbf{A}$. *Then there is a small category* $\mathbf{C}'$ *and a topology* J' *on* $\mathbf{C}'$ *so that* $\mathbf{A}^{\mathbf{S}}$ *is a category of sheaves of sets on* $\mathbf{C}'$ *with respect to* J'.

Proof. Let $\mathbf{B} = \mathrm{Sets}^{\mathbf{C}^{\mathrm{op}}}$, $\mathbf{R}$ the left exact idempotent triple corresponding to J, $\mathbf{T}$ the triple on $\mathbf{B}$ induced by $\mathbf{S}$, and $\hat{\mathbf{T}}$ the cocontinuous approximation to $\mathbf{T}$. Let $\mathbf{C}'$ be a category for which $\mathbf{B}^{\hat{\mathbf{T}}} \cong \mathrm{Sets}^{\mathbf{C}'^{\mathrm{op}}}$ (as in I). We have that $\mathbf{A}^{\mathbf{S}} \cong (\mathbf{B}^{\hat{\mathbf{T}}})^{\tilde{\mathbf{R}}}$, where $\tilde{\mathbf{R}}$ is a lifting of $\mathbf{R}$. Now the underlying functor from $\mathbf{B}^{\hat{\mathbf{T}}}$ to $\mathbf{B}$ is not only faithful but preserves and creates limits. Therefore, since $\mathbf{R}$ is a left exact idempotent triple, $\tilde{\mathbf{R}}$ must be also. We now let J' be the topology on $\mathbf{C}'$ corresponding to $\tilde{\mathbf{R}}$, and we are done. □

The referees of this paper have pointed out that Theorem 5 follows from Giraud's theorem (see [3, pp. 108–109]) in the following way. Since $\mathbf{S}$ is cocontinuous, the underlying functor $U: \mathbf{A}^{\mathbf{S}} \to \mathbf{A}$ creates both limits and colimits. From this one sees that $\mathbf{A}^{\mathbf{S}}$ is an exact category with limits, colimits, and disjoint universal sums. The free algebras in $\mathbf{A}^{\mathbf{S}}$ on the set of generators in $\mathbf{A}$ are easily seen to form a set of generators for $\mathbf{A}^{\mathbf{S}}$. Thus, by Giraud's theorem, $\mathbf{A}^{\mathbf{S}}$ is a topos, from which our Theorem 5 follows.

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