PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 47, Number 2, February 1975

CONVERGENCE SETS IN REFLEXIVE BANACH SPACES

BRUCE CALVERT

ABSTRACT. A closed linear subspace M of a reflexive Banach space X with X and X^* strictly convex is the range of a linear contractive projection iff $J(M)$ is a linear subspace of X^*. Hence the convergence set of a net of linear contractions is the range of a contractive projection if X and X^* are locally uniformly convex.

Let X be a Banach space over $\mathbf{C}$ or $\mathbf{R}$, and let (T_n) be a net of linear contractions on X. The convergence set for (T_n) is $\{x \in X: T_n x \to x\}$. Bernau [1] showed that if X is an L_p space, $p \in (1, \infty)$, then a convergence set is the range of a linear contractive projection. A simplification and generalization of this result follows from the characterisation of ranges of contractive linear projections of Theorem 1, and is given as Theorem 2 below.

Let S be a subset of X. Then the shadow of S is the set of x in X such that $T_n x \to x$ for every net of linear contractions on X such that $T_n y \to y$ for all y in S. Assuming X to be an L_p space, Bernau [1] showed that the shadow of S is the range of a contractive projection, and that if E is the range of a contractive projection, and E contains S, then E contains the shadow of S. This result holds generally, and is given as Corollary 2.

Theorem 3 considers finding the projection in terms of the net (T_n).

By a nearest point projection (on a subset K of a Banach space X) we mean a function Q taking x in X to a nearest point in K.

Lemma 1. *A set is the range of a linear contractive projection iff it is the nullspace of a linear nearest point projection.*

Proof. Q is a linear nearest point projection iff $I - Q$ is a linear contractive projection.

Theorem 1. *Let X be a strictly convex reflexive Banach space with strictly convex dual X^*. Let $J: X \to X^*$ be the duality map; $\|Jx\| = \|x\|$, $(Jx, x) = \|x\|^2$. Then a closed linear subspace M of X is the range of a*

Received by the editors November 1, 1973.

AMS (MOS) subject classifications (1970). Primary 46B99, 47B99.

Key words and phrases. Strictly convex, locally uniformly convex, contractive projection, convergence set, mean ergodic theorem.

linear contractive projection iff $J(M)$ is a linear subspace of X^*.

Proof. Suppose $J(M)$ is linear. Let Q be the nearest point projection on $J(M)^{\perp}$. (There exist nearest points since X is reflexive, and only one since X is strictly convex.) For $x \in X$, $Qx \in J(M)^{\perp}$ is defined by the property that for y in $J(M)^{\perp}$, $(J(x - Qx), y) = 0$. (The real part of J is the Gâteaux derivative of the function taking x to $\|x\|^2/2$, since X^* is strictly convex.) $J(M)$ is closed since J^{-1} is continuous from the strong to the weak topology (since X is reflexive and strictly convex). Hence, Qx is defined by $Qx \in (J(M))^{\perp}$ and $J(x - Qx) \in J(M)$, or $x - Qx \in M$. This shows Q is linear, for if $y \in X$, $y - Qy \in M$, $Qy \in (J(M))^{\perp}$, then $(x + y) - (Qx + Qy) \in M$, and $Qx + Qy \in (J(M))^{\perp}$, giving $Q(x + y) = Qx + Qy$, and similarly $Q(\alpha x) = \alpha Q(x)$.

Since $Qx = 0$ iff $x \in M$ the result follows from Lemma 1.

Conversely, suppose $M = R(P)$, the range of a contractive linear projection. If $m \in M$,

$$\|P^*Jm\| \leq \|Jm\| \leq \|m\|, \quad \text{and} \quad (P^*Jm, m) = (Jm, Pm) = \|m\|^2,$$

giving $P^*Jm = Jm$, since X^* is strictly convex. Hence, $J(M) \subset R(P^*)$. Replacing P by P^* and J by J^{-1} (since X is strictly convex), $J^{-1}R(P^*) \subset M$, giving $J(M) = R(P^*)$, completing the proof.

Corollary 0. *Let X be a reflexive Banach lattice with X and X^* strictly convex. Then a closed subspace M is the range of a positive linear contractive projection iff JM is a linear subspace and sublattice of X^* iff JM is a linear subspace and M is a sublattice.*

Proof. Let P be a positive linear contractive projection. Since P is positive, for x in X, $P(x^+) \geq (Px)^+$. Replacing x by Px gives $P((Px)^+) \geq (Px)^+$. Let y be a convex combination of $P((Px)^+)$ and $(Px)^+$. Since X is a Banach lattice, $\|y\| \geq \|(Px)^+\|$. Since $\|P\| = 1$, the opposite inequality holds. By strict convexity, $P((Px)^+) = (Px)^+$, which implies M is a sublattice. The argument applied to P^* gives $J(M)$ a sublattice.

Suppose JM is a linear subspace and M is a sublattice. For x in M and y in $J(M)^{\perp}$, $x^+ \in M$, giving

$$\|x^+\|^2 = (J(x^+), x) = (J(x^+), x + y) \leq (J(x^+), (x + y)^+) \leq \|x^+\| \, \|(x + y)^+\|.$$

If $x + y \leq 0$, then $x \leq 0$. Since $X = M + J(M)^{\perp}$, the linear contractive projection on M is positive.

Corollary 1. *Let X be as in Theorem 1. Let $(M_i)_{i\in I}$ be a net of ranges of linear contractive projections. Then $M = \bigcup_j \bigcap_{i\geq j} M_i$ is the range of a linear contractive projection.*

Proof. $J(\bigcup_j(\bigcap_{i\geq j} M_i)) = \bigcup_j \bigcap_{i\geq j} J(M_i)$ is a linear subspace since each $J(M_i)$ is.

Theorem 2. *Let X be a reflexive Banach space, with X and X^* locally uniformly convex. Then convergence sets are ranges of linear contractive projections.*

Proof. Let (T_n) be a net of contractions with convergence set M. If $m \in M$, $(T_n^* Jm, m) \longrightarrow \|m\|^2$, and $\|T_n^* Jm\| \leq \|m\|$, giving $T_n^* Jm \longrightarrow Jm$ since X^* is locally uniformly convex. (If Y is reflexive and locally uniformly convex, then given $y \neq 0$, and $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x\| \leq \|y\|$, and $\|x - y\| \geq \epsilon$, then $\|(x + y)/2\| \leq \|y\|(1 - \delta)$. Hence if a subsequence $T_{n'}^* Jm \longrightarrow y$ weakly, then $\|y\| = \|m\|$. Since $\|T_{n'}^* Jm\| \longrightarrow \|y\|$, $\|(T_{n'}^* Jm + y)/2\| \longrightarrow \|y\|$, which implies $T_{n'}^* Jm \longrightarrow y$. Since y satisfies the inequalities defining Jm, $y = Jm$.) Hence, $M^* \supset J(M)$ where $M^* = \{f \in X^*\colon T_n^* f \longrightarrow f\}$. Similarly, $M \supset J^{-1}(M^*)$, giving equality. The result follows by Theorem 1.

Corollary 2. *Let X be as in Theorem 2. Let $S \subset X$. Then the shadow of S is the smallest convergence set containing S.*

Proof. By definition, the shadow of S is the intersection of all convergence sets containing S, which is a convergence set by Theorem 2 and Corollary 1.

Corollary 3. *Let X be as in Theorem 1 and (T_n) a net of linear contractions. Then $\{x\colon T_n x \longrightarrow x\}$, the weak convergence set, is the range of a linear contractive projection.*

Proof. By the proof of Theorem 2.

Lemma 2. *Let X be a reflexive Banach space, with X and X^* strictly convex. If M is the range of a linear contractive projection P, it is the range of only one.*

Proof. By Theorem 1, $I - P$ is the nearest point projection on $N(P) = R(P^*)^\perp = (JM)^\perp$.

Lemma 3. *Let X be a reflexive Banach space with X and X^* strictly convex. Given a finite set $(T_n)_{n\in F}$ of linear contractions, then the linear contractive projection on $\{x\colon T_n x = x$ for n in $F\}$ is $\lim_{p\to\infty}\lim_{k\to\infty} A_{p,k}^F$*

(in the strong operator topology), where

$$A^F_{p,k} = \frac{1}{p} \sum_{j=0}^{p-1} \left(\prod_{n \in F} \frac{1}{k} \sum_{i=0}^{k-1} T^i_n \right)^j,$$

and the product can be taken in any order.

Proof. For n in F, let P_n be the linear contractive projection on $N(I - T_n)$, unique by Lemma 2. By the mean ergodic theorem, $T_n(k) = (1/k)\sum_{i=0}^{k-1} T^i_n$ converges to P_n.

By induction we show $\Pi_{n \in I} P_n x = x$, where $I \subseteq F$, implies $P_n x = x$ for n in I. Suppose it is true for m elements in I; then for n in F, suppose $P_n \Pi_{i \in I} P_i x = x$. If y is a convex linear combination of x and $\Pi_{i \in I} P_i x$, then $P_n y = x$, giving $\|y\| = \|x\|$. By strict convexity, $x = \Pi_{i \in I} P_i x$, giving $P_i x = x$ for x in I by the inductive hypothesis, and hence $P_n x = x$. Hence,

$$\bigcap_{n \in F} N(I - T_n) = \bigcap_n N(I - P_n) = N\left(I - \prod_{n \in F} P_n\right).$$

By the mean ergodic theorem, the nonexpansive projection on this set is $\lim_{p \to \infty} (1/p) \sum_{j=0}^{p-1} (\Pi P_n)^j$. By continuity of multiplication of operators in the strong topology, we can take the limits outside, giving the formula.

Lemma 4. *Let X be a reflexive Banach space, and $A = (A_n)_{n \in S}$ a net of bounded linear operators on X, $\|A_n\| \leq M$ for all n. Define $N(A) = \{x: A_n x \to 0\}$ and $R(A) = \{y$: there exists a subnet $(A_{n(m)})_{m \in T}$ of A, a bounded set of X, $\{y_{n(m)}: m \in T\}$, and for N in T there is a set of positive numbers α^N_m for finitely many $m \geq N$ in T, $\Sigma_m \alpha^N_m = 1$, and $y = \lim_N \Sigma_m \alpha^N_m A_{n(m)} y_{n(m)}\}$.*

Then by defining $A^* = (A^*_n)$, we have $R(A)^\perp = N(A^*)$.

Proof. Take $f \in N(A^*)$, $y \in R(A)$, $y = \lim_N \Sigma_m \alpha^N_m A_{n(m)} y_{n(m)}$, where $\|y_{n(m)}\| \leq K$ for all m. Then

$$(f, y) = \lim\left(f, \sum_m \alpha^N_m A_{n(m)} y_{n(m)}\right) = \lim\left(A^*_{n(m)} f, \sum_m \alpha^N_m y_{n(m)}\right)$$
$$\leq \overline{\lim} \|A^*_{n(m)} f\| K = 0.$$

Suppose instead that $f \in R(A)^\perp$. We wish to show that if T is a cofinal subset of S, then $(A^*_n f)_{n \in T}$ has a subnet converging to zero. Take $y_n =$

$J^{-1}A_n^*f$ for n in T. By weak compactness, there is a weak cluster point y for $(A_n y_n)_{n \in T}$. For N in T, y is in the weak closure of $\{A_p y_p : p \in T, p \geq N\}$, and hence the strong closure of its convex hull. Thus for U a neighborhood of y, we can take $\alpha_p^{N,U} \geq 0$, for $p \geq N$, p in T, nonzero for only finitely many p, $\Sigma_p \alpha_p^{N,U} = 1$, so that $\Sigma_p \alpha_p^{N,U} A_p y_p \in U$. Let Q be the directed set of neighborhoods of y; then for (p, U) in $T \times Q$, putting $A_{(p,U)} = A_p$ gives $(A_{p,U})_{(p,U) \in T \times Q}$ a subnet of $(A_p)_{p \in S}$ and

$$y = \lim_{(N,U) \in T \times Q} \sum_p \alpha_p^{N,U} A_p y_p,$$

giving $y \in R(A)$. But

$$0 = \lim_{(N,U)} \left(f, \sum_p \alpha_p^{N,U} A_p y_p \right) = \lim_{(N,U)} \sum_p \alpha_p^{N,U} \|A_p^* f\|^2 \geq \underline{\lim}_{p \in T} \|A_p^* f\|^2,$$

completing the proof.

Theorem 3. *Let X be a reflexive Banach space with X and X^* strictly convex. Let $(T_n)_{n \in S}$ be a net of contractions such that $x = \lim T_n x$ implies $x = T_n x$ eventually. Then the convergence set M is the range of the linear contractive projection*

$$A = \lim_N \lim_{F \in Q_N} \lim_{p \to \infty} \lim_{k \to \infty} A_{p,k}^F,$$

where Q_N is the set of finite subsets of the set of elements p of S, $p \geq N$, directed under inclusion. $A_{p,k}^F$ is defined in Lemma 3.

Proof. $M = \bigcup_{N \in S} \bigcap_{n \geq N} N(I - T_n)$. By Corollary 1, M is the range of a linear contractive projection. Set $I - T = (I - T_n)_{n \in S}$. By Lemma 4, $X = M + \operatorname{cl} R(I - T)$. Given $\epsilon > 0$, for x in M and z in $\operatorname{cl} R(I - T)$, take y in $R(I - T)$, $\|y - z\| < \epsilon/3$, let $y = \lim_N \Sigma_m \alpha_m^N (I - T_{n(m)}) y_{n(m)}$, where $\|y_{n(m)}\| \leq K$, take N, α_m^N, such that $x \in N(I - T_n)$ for $n \geq N$ and $\|y - \Sigma \alpha_m^N (I - T_{n(m)}) y_{n(m)}\| < \epsilon/3$. Take F_N the support of α_m^N, $F = n(F_N)$, and set

$$A_{p,k}^F = \frac{1}{p} \sum_{j=0}^{p-1} \left(\prod_{n \in F} T_n(k) \right)^j,$$

where $T_n(k) = (1/k) \Sigma_{i=0}^{k-1} T_n^i$, and some order is chosen in the product. Choose p and q by Lemma 3 so that $\|A_{p,k}^F \Sigma \alpha_m^N (I - T_{n(m)}) y_{n(m)}\| < \epsilon/3$. Then

$$\begin{aligned}\|A^F_{p,k}(x+z)-x\| &\leq \|A^F_{p,k}(z-y)\| + \|A^F_{p,k}(y-\sum \alpha^N_m(I-T_{n(m)})y_{n(m)})\| \\ &\quad + \|A^F_{p,k}\sum \alpha^N_m(I-T_{n(m)})y_{n(m)}\| \\ &\leq \epsilon/3+\epsilon/3+\epsilon/3,\end{aligned}$$

proving the claim.

REFERENCE

1. S. J. Bernau, *Theorems of Korovkin type for L_p spaces*, Pacific J. Math. (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AUCKLAND, AUCKLAND, NEW ZEALAND