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NORMAL AND HERMITIAN COMPOSITION OPERATORS

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ABSTRACT. Let C_ϕ be a composition operator on $L^2(\lambda)$. Some conditions under which C_ϕ is an isometry and Hermitian are investigated in this paper. Some study of normal composition operators is also made.

1. **Introduction.** This paper is a continuation of [3]. Let λ be a σ-finite measure on a set X, and let ϕ be a measurable transformation on X into itself. Then define a composition transformation C_ϕ on $L^2(\lambda)$ as $C_\phi f = f \circ \phi$ for every f in $L^2(\lambda)$. If C_ϕ is bounded, then it is called a composition operator. In [3] we have characterized all quasinormal composition operators. Our purpose in this paper is to report on normal and Hermitian composition operators. In §2 we shall investigate all isometric composition operators, and we shall also prove that in certain cases quasinormality implies normality. §3 is devoted to the characterization of Hermitian composition operators. This class of operators has been characterized by Ridge [2]. Since our proofs and approaches are entirely different from Ridge's, they are given here.

2. **Isometric and normal composition operators.** If $\phi: X \to X$ is a measurable transformation, then $\lambda\phi^{-1}$ is a measure on X. Also if C_ϕ is a composition operator, then $\lambda\phi^{-1}$ is absolutely continuous with respect to λ. Let us denote the Radon-Nikodym derivative of $\lambda\phi^{-1}$ by f_0. Ridge [2] proves that if $f_0 = 1$ (a.e.), then C_ϕ is an isometry. This condition is also necessary, as the following theorem shows.

Theorem 1. *Let C_ϕ be a composition operator on $L^2(\lambda)$. Then a necessary and sufficient condition that C_ϕ be an isometry is that $f_0 = 1$ (a. e.).*

Proof. The proof is a trivial consequence of the relation $C_\phi^* C_\phi = M_{f_0}$, where M_{f_0} is the multiplication operator induced by f_0.

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It is well known that every normal operator is quasinormal, but the converse is not true. However, in the case of composition operators induced by one-to-one (a.e.) measurable transformations, the converse is also true. This is shown in the following theorem.

Theorem 2. *Let C_ϕ be a composition operator on $L^2(\lambda)$ induced by a one-to-one (a.e.) transformation ϕ. Then C_ϕ is quasinormal if and only if it is normal.*

Proof. First we claim that if ϕ is one-to-one (a.e.), then the range of C_ϕ is dense. We shall outline the proof as follows. Since ϕ is a one-to-one (a.e.), there exists a measurable transformation ψ such that $(\psi \circ \phi)(x) = x$ (a.e.). If $\lambda(X) < \infty$, then the range of C_ϕ is obviously dense as it contains all characteristic functions, and hence all simple functions. So let $\lambda(X) = \infty$. Since λ is σ-finite, we can write $X = \bigcup_{i=1}^{\infty} X_i$, where X_i's are disjoint and $\lambda(X_i) < \infty$ for every i. Let χ_E (the characteristic function of the set E) be in $L^2(\lambda)$. Then

$$\chi_{\psi^{-1}(E)} = \sum_{i=1}^{\infty} \chi_{F_i},$$

where $F_i = \psi^{-1}(E) \cap X_i$. Consider the sum $\sum_{i=1}^{\infty} \chi_{F_i} \circ \phi$. If $x \in X$, it is clear that $\phi(x) \in F_i$ for some i iff $\phi(x) \in \psi^{-1}(E)$. Hence the above sum converges to χ_E (a.e.). By the Lebesgue dominated convergence theorem, it converges to χ_E in L^2-norm, and hence χ_E is in the closure of the range of C_ϕ. This is enough to show that the range of C_ϕ is dense in $L^2(\lambda)$.

Now we shall prove the theorem. Suppose C_ϕ is normal; then it follows trivially that it is quasinormal. On the other hand, suppose C_ϕ is quasinormal. We know that $C_\phi^* C_\phi = M_{f_0}$. Hence

$$C_\phi C_\phi^* C_\phi = C_\phi M_{f_0},$$

and since C_ϕ is quasinormal, we get $C_\phi C_\phi^* C_\phi = M_{f_0} C_\phi$. This shows that $C_\phi C_\phi^*$ and M_{f_0} agree on the range of C_ϕ, which is dense in $L^2(\lambda)$. Hence

$$C_\phi C_\phi^* = M_{f_0} = C_\phi^* C_\phi.$$

This proves that C_ϕ is normal.

3. Hermitian composition operators. Hermitian composition operators are characterized by Ridge [2], where he uses the techniques depending on the spectrum of the operator. In Theorem 3 a different characterisation, using a simple and different method, is given.

Theorem 3. *Let C_ϕ be a composition operator on $L^2(\lambda)$. Then C_ϕ is Hermitian if and only if $f_0 = 1$ (a.e.) and $(\phi \circ \phi)(x) = x$ (a.e.).*

Proof. Suppose C_ϕ is Hermitian. Then $C_\phi = C_\phi^*$. Since $C_\phi^* C_\phi = M_{f_0}$, we get $C_{\phi\circ\phi} = C_\phi C_\phi = M_{f_0}$. We can write $X = \bigcup_{n=1}^\infty E_n$, where $\lambda(E_n) < \infty$ for each n, and $E_n \subset E_m$ if $m > n$. If $f_n = X_{E_n}$, then we obtain $C_{\phi\circ\phi} f_n = M_{f_0} f_n$. Equivalently, $X_{(\phi\circ\phi)^{-1}(E_n)} = f_0 \cdot X_{E_n}$. Since $\bigcup_{n=1}^\infty (\phi\circ\phi)^{-1}(E_n) = X$, we can conclude that $f_0 = 1$ (a.e.) and also $(\phi\circ\phi)(x) = x$ (a.e.).

Conversely, suppose $f_0 = 1$ (a.e.) and $(\phi \circ \phi)(x) = x$ (a.e.). Then

$$C_\phi^* C_\phi = M_{f_0} = I \quad \text{(the identity operator)}.$$

Multiplying both sides on the right by C_ϕ we get

$$C_\phi^* C_\phi C_\phi = C_\phi, \quad C_\phi^* C_{\phi\circ\phi} = C_\phi, \quad C_\phi^* = C_\phi.$$

This shows that C_ϕ is Hermitian.

Corollary. *If C_ϕ is a composition operator on $L^2(\lambda)$, then C_ϕ is a projection if and only if $C_\phi = I$.*

Corollary. *C_ϕ is Hermitian implies C_ϕ is an isometry.*

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