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EQUIDISTANT SETS AND THEIR CONNECTIVITY PROPERTIES

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ABSTRACT. If A and B are nonvoid subsets of a metric space (X, d), the set of points $x \in X$ for which $d(x, A) = d(x, B)$ is called the equidistant set determined by A and B. Among other results, it is shown that if A and B are connected and X is Euclidean n-space, then the equidistant set determined by A and B is connected.

1. **Introduction.** In the Euclidean plane the set of points equidistant from two distinct points is a line, and the set of points equidistant from a line and a point not on it is a parabola. It is less well known that ellipses and single branches of hyperbolae admit analogous definitions. The set of points equidistant from two nested circles is an ellipse with foci at their centres. The set of points equidistant from two disjoint disks of different sizes is that branch of an hyperbola with foci at their centres which opens around the smaller disk. These classical examples prompt us to inquire further about the properties of sets which can be realised as equidistant sets.

The most general context in which this study is meaningful is that of a metric space (X, d). If A is a nonvoid subset of X, and x is a point of X, then the distance from x to A is defined to be

$$d(x, A) = \inf\{d(x, a):\ a \in A\}.$$

If A and B are both nonvoid subsets of X then the equidistant set determined by A and B is defined to be

$$\{A = B\} = \{x:\ d(x, A) = d(x, B)\}.$$

This notation admits convenient generalization to $\{A \le B\} = \{x: d(x, A) \le$

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$d(x, B)\}$ and $\{A < B\} = \{x: d(x, A) < d(x, B)\}$.

The properties of equidistant sets which can be established in a general metric space are discussed in the next section. Then the metric space is specialized to Euclidean n-space and a more detailed analysis is made of their topological properties.

2. **Equidistant sets in metric spaces.** In terms of the point-set distance, the closure of a set A is just $\overline{A} = \{x: d(x, A) = 0\}$. It follows easily that $d(x, A) = d(x, \overline{A})$ and, therefore, that $\{A = B\} = \{\overline{A} = \overline{B}\}$. In general, $\{A = B\} \supset \overline{A} \cap \overline{B}$ because these points have distance zero to both sets. If $\overline{A} \cup \overline{B} = X$, then $\{A = B\} = \overline{A} \cap \overline{B}$ because other points have a positive distance to one set and zero distance to the other. If $\overline{A} = \overline{B}$, then $\{A = B\} = X$.

The function $d_A: X \to R$ defined by $d_A(x) = d(x, A)$ is Lipschitz and therefore continuous. The sets $\{A = B\}$, $\{A \leq B\}$ and $\{A < B\}$ are, respectively, the inverse images of $\{0\}$, $(-\infty, 0]$ and $(-\infty, 0)$ under the continuous function $d_A - d_B$. It follows that $\{A = B\}$ and $\{A \leq B\}$ are closed while $\{A < B\}$ is open. Trivially, $\{A \leq B\} = \{A < B\} \cup \{A = B\}$, but it is not generally true that $\{A = B\}$ is the boundary of $\{A < B\}$.

Not only are equidistant sets always closed, but, conversely, any nonvoid closed set A is an equidistant set. This is true because $\{A = X\} = \overline{A} \cap \overline{X} = A$. The connectivity of X is related to the possibility of the void set being an equidistant set.

Theorem 1. *The metric space X is connected if and only if equidistant sets in X are never void.*

Proof. If X is not connected, $X = A \cup B$ for some pair of nonvoid closed disjoint sets A and B. Then $\{A = B\} = A \cap B = \emptyset$.

It remains to show that this cannot happen when X is connected. If A and B are nonvoid subsets, then $\{A \leq B\}$ is nonvoid because it contains A. Thus $X = \{A \leq B\} \cup \{B \leq A\}$ is the union of nonvoid closed sets, and if X is connected, their intersection $\{A \leq B\} \cap \{B \leq A\} = \{A = B\}$ must be nonvoid.

3. **Equidistant sets in Euclidean n-space.** If E is an m-dimensional flat in Euclidean n-space, the distance from a general point to a point of E may be determined from distances measured in E and perpendicular to E. It follows that if A and B are subsets of E, then $\{A = B\}$ is a cylinder based on $\{A = B\} \cap E$.

If A is nonvoid closed set in Euclidean n-space, then $d(x, A) = d(x, a_0)$ for some point a_0 in A. For if $d(x, A) = \delta$, the closed ball of radius $\delta + 1$ about x meets A in a compact set C, and $d(x, A) = \inf\{d(x, a): a \in C\}$ is realized as a minimum. This useful remark is instrumental in proving

Theorem 2. *If A and B are nonvoid subsets of Euclidean n-space such that $\overline{A} \cap \overline{B} = \emptyset$, then $\{A = B\}$ has void interior and is the common boundary of $\{A < B\}$ and $\{B < A\}$.*

Proof. Let $e \in \{A = B\}$ where A and B satisfy the conditions of the theorem. Then $d(e, A) = d(e, B) = \delta > 0$. The open ball of radius δ about e contains no points of $\overline{A}$ or $\overline{B}$ while its bounding sphere contains at least two distinct points $a_0 \in \overline{A}$ and $b_0 \in \overline{B}$. If a point x on the half open radial segment $(e, a_0]$ has distance $\delta_1 < \delta$ from a_0, then the closed ball of radius δ_1 about x meets $\overline{A} \cup \overline{B}$ at the single point a_0. It follows that $(e, a_0] \subset \{A < B\}$, and similarly that $(e, b_0] \subset \{B < A\}$. This gives the theorem.

As an application of the theorem, consider a finite or denumerably infinite family A_i $(i \in I)$ of nonvoid subsets of n-space which satisfy $\overline{A}_i \cap \overline{A}_j = \emptyset$ for $i \neq j$. Then for "most" points x in the n-space, the numbers $d(x, A_i)$ $(i \in I)$ are all different. The reason for this surprising fact is that an equality occurs only if x lies in some $\{A_i = A_j\}$. But the theorem shows that these sets are nowhere dense and so a countable union of them is only of first Baire category.

We are prompted to ask for a complete description of the equidistant sets that are determined after the fashion of Theorem 2. However, this remains an open question, and so we return now to the issue of connectivity. The main theorems stated below do not make use of the hypothesis $\overline{A} \cap \overline{B} = \emptyset$.

Theorem 3. *In Euclidean n-space if $A = \{a\}$ is a singleton and B an arbitrary nonvoid set, then $\{A = B\}$ is either connected or the union of two parallel hyperplanes. The second case arises if and only if $a \notin \overline{B}$ and B is a subset of a line through a which meets both of the rays into which a divides it.*

Theorem 4. *In Euclidean n-space if A and B are nonvoid connected sets, then $\{A = B\}$ is connected.*

It is not true that if A and B are path connected then $\{A = B\}$ is path connected. An amusing counterexample in the plane is given by two interlocked combs. Let

$$A = \{(x, 1): x \geq 0\} \cup \{(x, y): x = 1/n, -1 + 1/n \leq y \leq 1, n = 1, 3, 5, \cdots\}$$

and

$$B = \{(x, -1): x \geq 0\} \cup \{(x, y): x = 1/n, -1 \leq y \leq 1 - 1/n, n = 2, 4, 6, \cdots\}.$$

Then $\{A = B\}$ is the closed halfplane, $x \leq 0$, together with a curve resembling the graph of $y = \sin 1/x$, $x > 0$, but made up of segments of straight lines and parabolas.

Before starting to prove Theorems 3 and 4, let us notice how their analogues break down when the metric space is changed from Euclidean n-space to the circle $\{e^{i\theta}: 0 \leq \theta < 2\pi\}$. Here, if $A = \{1\}$ and $B = \{-1\}$, the equidistant set $\{A = B\} = \{i, -i\}$ which is not connected. As the proofs will indicate, part of the problem is that the circle is not simply connected. But more is at stake because the counterexample still stands if the circle is replaced by its simply connected subset $\{e^{i\theta}: 0 \leq \theta \leq 3\pi/2\}$.

The proofs of Theorems 3 and 4 are developed in the following sections.

4. **Proof of Theorem 3.** In this section $A = \{a\}$ is a singleton and B is an arbitrary nonvoid subset of n-space.

Lemma 1. *The set $\{\{a\} \leq B\}$ is convex and therefore connected.*

Proof. The set $\{\{a\} \leq B\}$ is the intersection of the closed halfspaces $\{\{a\} \leq \{b\}\}$ with $b \in \overline{B} - \{a\}$.

Lemma 2. *If $a \in \overline{B}$, the equidistant set $\{\{a\} = B\}$ is connected.*

Proof. If $a \in \overline{B}$, $d(x, B) \leq d(x, a)$ and so $\{\{a\} = B\} = \{\{a\} \leq B\}$. The result follows from Lemma 1.

For the rest of the proof of Theorem 3 it is possible to assume that $a \notin \overline{B}$. Let R denote an arbitrary closed ray issuing from the point a, and $H(R)$ the open halfspace bordering a with R as inward pointing normal. The ray R meets the equidistant set exactly when $H(R)$ meets B and then in a single point $e(R) \neq a$.

We shall prove (Lemma 3) that if the rays from a are given the topology of a sphere about a, then the mapping $R \to e(R)$ is continuous. Also (Lemma 4) for dimension $n \geq 2$ if every line through a meets the equidistant set, then the domain of this mapping is connected. These two results prove that the equidistant set is connected.

On the other hand, if there is a line L through a which misses the equidistant set, then the set B must be in the $(n-1)$-dimensional flat E through a perpendicular to L. Thus the equidistant set is a cylinder based on $\{\{a\} = B\} \cap E$ and will be connected if and only if this $(n-1)$-dimensional equidistant set is connected. An inductively chosen sequence of lines and orthogonal flats may terminate at dimension $m \geq 2$ in a situation where a connected equidistant set is guaranteed by the argument of the preceding paragraph. Then $\{\{a\} = B\}$ is a connected cylinder. Alternatively the induction may prove that B is a subset of a line E_1 through a. Then $\{\{a\} = B\}$ is either a single hyperplane or a pair of parallel hyperplanes depending on whether B meets just one or both of the rays into which a divides E_1.

Lemma 3. *The mapping $R \to e(R)$ is continuous.*

Proof. Let R_0 meet the equidistant set at $e(R_0)$. Let $d(e(R_0), B) = d(e(R_0), b_0)$ for $b_0 \in \overline{B}$. Then the hyperplane $\{\{a\} = \{b_0\}\}$ provides an outer bound for points $e(R)$ on rays R near R_0.

Since $a \notin \overline{B}$ there is a closed ball about a lying in $\{\{a\} < B\}$. Since $\{\{a\} \leq B\}$ is convex, it contains the cone of tangents from $e(R_0)$ to this ball. The cone provides an inner bound for points $e(R)$ on rays R near R_0.

Lemma 4. *In dimension $n \geq 2$, if every line through a meets the equidistant set, then the set of all rays which meet it is connected.*

Proof. If R_1 and R_2 fail to meet the equidistant set they meet at an angle $\theta < \pi$. If R is a ray in the angle θ, then because $H(R) \subset H(R_1) \cup H(R_2)$, it fails to meet B, and consequently R fails to meet the equidistant set. Thus on any great circle the rays which fail to meet the equidistant set lie on an arc, and those which do meet it on the complementary arc. If R_0 is a fixed ray which meets the equidistant set, every other ray which does so may be joined to R_0 by a great circular arc of rays of the same type.

5. **Proof of Theorem 4.** If Euclidean n-space $E_n = A \cup B$, where A and B are nonvoid closed and connected, then $A \cap B$ is nonvoid and closed, but is it connected? The remarks about equidistant sets in metric spaces show that with the preceding conditions $\{A = B\} = A \cap B$, so the answer must be yes if Theorem 4 is true. The next lemma reduces Theorem 4 to this special case by allowing us to replace given connected sets A and

B by the sets $\{A \leq B\}$ and $\{B \leq A\}$ from which $\{A = B\}$ may be determined as $\{A \leq B\} \cap \{B \leq A\}$.

Lemma 5. *If A and B are nonvoid subsets of E_n and A is connected, then $\{A \leq B\}$ is connected.*

Proof. Because point-set distances are realized in closed sets, $\{A \leq B\}$ is the union of the sets $\{\{a\} \leq B\}$ with $a \in \bar{A}$. By Lemma 1, $\{\{a\} \leq B\}$ is connected. Since $\bar{A}$ is connected and $\{\{a\} \leq B\} \cap \bar{A} \supset \{a\}$, it is possible to include $\bar{A}$ in the union and deduce that $\{A \leq B\}$ is connected from the standard result on unions.

A result analogous to the one required to complete the proof of Theorem 4 is provided by

Lemma 6. *If $E_n = Y \cup Z$ where Y and Z are nonvoid, open and connected, then $Y \cap Z$ is connected.*

Proof. Since Y and Z are open, the Mayer-Vietoris sequence of homological modules is exact. The tail of this sequence is

$$\to H_1(E_n) \to H_0(Y \cap Z) \to H_0(Y) \oplus H_0(Z) \to H_0(E_n) \to 0.$$

Since E_n is contractible, its first homology group is trivial and the sequence reduces to

$$0 \to H_0(Y \cap Z) \to H_0(Y) \oplus H_0(Z) \to H_0(E_n) \to 0.$$

Open connected subsets of E_n are path connected, and the dimension of H_0 counts the number of these components. It follows that $Y \cap Z$ is path connected.

The proof of Theorem 4 is completed with

Lemma 7. *If $E_n = A \cup B$, where A and B are nonvoid, closed and connected, then $A \cap B$ is connected.*

Proof. If $A \cap B$ is not connected, then $A \cap B = C_1 \cup C_2$ for nonvoid disjoint closed sets C_1 and C_2. Since E_n is normal, there are disjoint open sets U_i such that $C_i \subset U_i$ $(i = 1, 2)$. Replace these open sets by open sets $V_i \subset U_i$ which can be written as the union of open balls centred in C_i $(i = 1, 2)$. Then define $V = V_1 \cup V_2$ and write $Y = A \cup V$ and $Z = B \cup V$.

Since $A \cap B \subset V$, $Y = B' \cup V$ and is open. Since Y is equal to A together with certain open balls meeting A, it is connected. Similarly Z

is open and connected. By Lemma 6, $Y \cap Z = V$ is connected. But $V = V_1 \cup V_2$ is the union of nonvoid disjoint open sets. This contradiction completes the proof.

6. **Addendum.** H. Bell and S. K. Kaul have independently suggested an alternative proof of Lemma 5. They define $f: E_n \to \bar{A}$ such that $d(x, A) = d(x, f(x))$ and remark that if $x \in \{A \leq B\}$ then the closed segment $[x, f(x)] \subset \{A \leq B\}$. This approach may be compared with the proof of Theorem 2.

S. Ferry has kindly drawn my attention to a number of related references. In [1] H. Bell proves that if A and B are disjoint closed connected subsets of E_2 then $\{A = B\}$ is a 1-manifold. He also presents a counterexample to show that the analogous result is false in E_3. In [2]–[4] other authors investigate the distance function $d_A: E_n \to R$ to see when its level sets are $(n-1)$-manifolds.

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