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WELL-DISTRIBUTED SEQUENCES WITH RESPECT TO SYSTEMS OF CONVEX SETS

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ABSTRACT. A theorem of W. M. Schmidt concerning the existence of sequences which are extremely well distributed with respect to suitable convex sets is generalized. We prove the existence of sequences which are simultaneously well distributed with respect to suitable systems of convex sets. The proof depends on combinatorial results dealing with the distribution of sequences in finite and countable sets.

1. **Introduction.** For $s \geq 2$, let $\mathbf{U}^s = \{(u_1, \cdots, u_s) \in \mathbf{R}^s\colon 0 \leq u_i < 1$ for $1 \leq i \leq s\}$ be the s-dimensional half-open unit cube. We consider a sequence $\mathbf{x}_1, \mathbf{x}_2, \cdots$ of points in $\mathbf{U}^s$. For a positive integer n and a Lebesgue measurable subset S of $\mathbf{U}^s$, let $Z(n, S)$ be the number of r, $1 \leq r \leq n$, with $\mathbf{x}_r \in S$. We define the local discrepancy $D(n, S) = |Z(n, S) - n\lambda(S)|$, where λ denotes the s-dimensional Lebesgue measure, and $E(S) = \sup_n D(n, S)$. In a recent paper, W. M. Schmidt has shown the following remarkable theorem which is connected with the notion of isotropic discrepancy (see [1, Chapter 2]) and results on irregularities of distribution in [6], [7], [8].

Theorem 1 (Schmidt [8]). *For any $s \geq 2$, there exists a sequence $\mathbf{x}_1$, $\mathbf{x}_2, \cdots$ in $\mathbf{U}^s$ such that for every μ with $0 \leq \mu \leq 1$ there is a convex subset S of $\mathbf{U}^s$ satisfying $\lambda(S) = \mu$ and $E(S) \leq \frac{1}{2}$.*

In this note, we prove a generalization of Theorem 1 to systems of convex sets. The basic idea is to combine Schmidt's method with some deep results in combinatorial theory pertaining to the distribution of sequences in finite and countable sets. Our final result is as follows.

Theorem 2. *For any $s \geq 2$, there exists a sequence $\mathbf{x}_1, \mathbf{x}_2, \cdots$ in $\mathbf{U}^s$*

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satisfying the following property: for every integer $k \geq 2$ *and any numbers* $\mu_1, \cdots, \mu_k$ *in* $[0, 1]$ *with* $\sum_{j=1}^{k} \mu_j = 1$, *and also for any sequence* $\mu_1, \mu_2, \cdots$ *of numbers in* $[0, 1]$ *with* $\sum_{j=1}^{\infty} \mu_j = 1$, *there are convex subsets* $S_1, \cdots, S_k$ *(resp.* $S_1, S_2, \cdots$*) of* $\mathbf{U}^s$ *with* $\lambda(S_j) = \mu_j$ *for* $1 \leq j \leq k$ *(resp.* $1 \leq j < \infty$*) and* $E(S_j) \leq 1 - 1/(2k-2)$ *for* $1 \leq j \leq k$ *(resp.* $E(S_j) \leq 1$ *for* $1 \leq j < \infty$*), and such that every point* $\mathbf{x}_n$ *of the sequence lies in a unique* S_j.

We note that the case $k = 2$ in Theorem 2 yields Theorem 1 (see also Remark 2). To avoid a trivial case, we remark that if $\mu_j = 0$ for some j, we may take the corresponding S_j to be the empty set. Thus, in the sequel, we can assume that $0 < \mu_j < 1$ for all j.

2. **Some combinatorial lemmas.** In this section, we collect some useful facts concerning the distribution of sequences in finite and countable sets. This subject was studied recently in a number of papers [2], [3], [4], [5], [9].

For $k \geq 1$, let $j_1, j_2, \cdots$ be a sequence of elements from the set $\mathbf{Z}_k = \{1, 2, \cdots, k\}$. Given integers $n \geq 1$ and $j \in \mathbf{Z}_k$, we define the counting function $A(n, j)$ to be the number of r, $1 \leq r \leq n$, with $j_r = j$.

Lemma 1_k (Meijer [2]). *For any* $k \geq 2$ *and any numbers* $\mu_1, \cdots, \mu_k$ *in* $(0, 1)$ *with* $\sum_{j=1}^{k} \mu_j = 1$, *there exists a sequence* $j_1, j_2, \cdots$ *in* $\mathbf{Z}_k$ *satisfying* $|A(n, j) - n\mu_j| \leq 1 - 1/(2k-2)$ *for all* $n \geq 1$ *and all* $j \in \mathbf{Z}_k$.

For a sequence $j_1, j_2, \cdots$ of elements from the set $\mathbf{Z}_\infty = \{1, 2, \cdots\}$, we define the counting function $A(n, j)$ in a like manner as above.

Lemma 1_∞ (Tijdeman [9]). *For any sequence* $\mu_1, \mu_2, \cdots$ *of numbers in* $(0, 1)$ *with* $\sum_{j=1}^{\infty} \mu_j = 1$, *there exists a sequence* $j_1, j_2, \cdots$ *in* $\mathbf{Z}_\infty$ *satisfying* $|A(n, j) - n\mu_j| \leq 1$ *for all* $n \geq 1$ *and* $j \geq 1$.

To combine the two cases, we write $\epsilon_k = 1/(2k-2)$ for $2 \leq k < \infty$ and $\epsilon_\infty = 0$. In order to unify the discussion, k may also attain the value ∞ from now on. We need a simple auxiliary result complementing the above two lemmas in the case of some μ_j being close to 1.

Lemma 2. *Let* $2 \leq k \leq \infty$, *let the* μ_j, $j \in \mathbf{Z}_k$, *be as in Lemma* 1_k, *and suppose that* $\frac{1}{2} < \mu_h < 1$ *for some* $h \in \mathbf{Z}_k$. *Let* m *be the largest integer such that* $\mu_h > 1 - 1/m$. *Then there exists a sequence* $j_1, j_2, \cdots$ *in* $\mathbf{Z}_k$ *with* $j_r = h$ *for* $1 \leq r \leq [m/2]$ *and* $|A(n, j) - n\mu_j| \leq 1 - \epsilon_k$ *for all* $n \geq 1$ *and all* $j \in \mathbf{Z}_k$.

Proof. According to Lemma 1_k, there exists a sequence $i_1, i_2, \cdots$ in

$\mathbf{Z}_k$ with $|A(n, j) - n\mu_j| \leq 1 - \epsilon_k$ for all $n \geq 1$ and all $j \in \mathbf{Z}_k$. In particular, we have

$$A([m/2] + 1, h) \geq ([m/2] + 1)\mu_h - 1 + \epsilon_k.$$

Using $m \geq 2$, it follows that

$$A([m/2] + 1, h) > ([m/2] + 1)(1 - 1/m) - 1 + \epsilon_k$$

$$\geq [m/2] - 1 + \epsilon_k \geq [m/2] - 1.$$

Since $A([m/2] + 1, h)$ is an integer, we arrive at $A([m/2] + 1, h) \geq [m/2]$. Thus, at most one of the i_r with $1 \leq r \leq [m/2] + 1$ can be different from h. It remains to consider the case that $i_r \neq h$ for some r with $1 \leq r \leq [m/2]$. We define a new sequence $j_1, j_2, \cdots$ in $\mathbf{Z}_k$ by setting $j_n = h$ for $1 \leq n \leq [m/2]$, $j_n = i_r$ for $n = [m/2] + 1$, and $j_n = i_n$ for $n > [m/2] + 1$. For $1 \leq n \leq r - 1$, and also for $n \geq [m/2] + 1$, the counting functions $A(n, j)$ of the old and the new sequence are identical. For $r \leq n \leq [m/2]$, the counting function of the new sequence satisfies

$$|A(n, h) - n\mu_h| = n(1 - \mu_h) < n/m \leq ½ \leq 1 - \epsilon_k,$$

and

$$|A(n, j) - n\mu_j| = n\mu_j < n/m \leq 1 - \epsilon_k$$

for $j \neq h$, since in this case $\mu_j < 1/m$. Therefore the new sequence enjoys all the required properties.

3. **Proof of Theorem 2.** We observe that it suffices to prove the theorem for $s = 2$. For if $\mathbf{x}_1 = (x_1, y_1)$, $\mathbf{x}_2 = (x_2, y_2)$, $\cdots$ is a sequence in $\mathbf{U}^2$ serving the desired purpose, then for $s \geq 3$ the sequence $\mathbf{x}_1', \mathbf{x}_2', \cdots$ in $\mathbf{U}^s$ defined by $\mathbf{x}_n' = (x_n, y_n, 0, \cdots, 0)$ for $n \geq 1$ will do. To see this, one just has to take for any k, $2 \leq k \leq \infty$, the convex subsets S_j, $j \in \mathbf{Z}_k$, of $\mathbf{U}^2$ associated with a possible choice of numbers μ_j, $j \in \mathbf{Z}_k$, according to Theorem 2, and to replace them by the convex subsets $S_j' = S_j \times \mathbf{U}^{s-2}$, $j \in \mathbf{Z}_k$, of $\mathbf{U}^s$.

We consider now the case $s = 2$. Let $t_1, t_2, \cdots$ be a decreasing sequence of real numbers with $0 < t_n \leq 1/(8n)$ for $n \geq 1$, and put

$$\mathbf{x}_n = (x_n, y_n) = (1 - \cos t_n, \sin t_n)$$

for $n \geq 1$. All the points $\mathbf{x}_n$ lie on the circle $(x - 1)^2 + y^2 = 1$. For a given k, $2 \leq k \leq \infty$, let the μ_j, $j \in \mathbf{Z}_k$, be numbers in the interval $(0, 1)$ with $\sum_{j \in \mathbf{Z}_k} \mu_j = 1$. According to Lemma 1_k, there exists a sequence $j_1, j_2, \cdots$

in $\mathbf{Z}_k$ with $|A(n, j) - n\mu_j| \le 1 - \epsilon_k$ for all $n \ge 1$ and all $j \in \mathbf{Z}_k$. In case $\frac{1}{2} < \mu_h < 1$ for some $h \in \mathbf{Z}_k$, we suppose also that the sequence $j_1, j_2, \cdots$ satisfies the additional condition in Lemma 2. For $j \in \mathbf{Z}_k$ and $n \ge 1$, let $F(n, j)$ be the set of those $\mathbf{x}_i$, $1 \le i \le n$, with $j_i = j$. We have of course $\operatorname{card} F(n, j) = A(n, j)$, and therefore

$$(1) \qquad |\operatorname{card} F(n, j) - n\mu_j| \le 1 - \epsilon_k \quad \text{for all } n \ge 1 \text{ and all } j \in \mathbf{Z}_k.$$

Moreover, the sets $F(n, j)$ and $F(n', j')$ are disjoint as soon as $j \ne j'$.

For fixed $j \in \mathbf{Z}_k$, we define a convex subset $G_1(j)$ of $\mathbf{U}^2$ by letting $G_1(j)$ be the convex hull of the points $\mathbf{x}_i$ appearing in some $F(n, j)$. It follows from (1) that $\lim_{n\to\infty} \operatorname{card} F(n, j) = \infty$. In addition, the sequence $\operatorname{card} F(n, j)$, $n = 1, 2, \cdots$, is nondecreasing and attains all positive integers as values. Let n_2 be the smallest positive integer with $\operatorname{card} F(n_2, j) = 2$. Then there exists n_1, $1 \le n_1 < n_2$, such that $\mathbf{x}_{n_1}, \mathbf{x}_{n_2} \in F(n_2, j) \subseteq G_1(j)$. It is easily seen that $\lambda(G_1(j))$ is at most the area of the triangle bounded by the y-axis, the line segment joining $\mathbf{x}_{n_1}$ and the origin, and the line joining $\mathbf{x}_{n_1}$ and $\mathbf{x}_{n_2}$. Hence we get

$$\lambda(G_1(j)) \le \tfrac{1}{2} y_{n_2} = \tfrac{1}{2} \sin t_{n_2} < \tfrac{1}{2} t_{n_2} \le 1/(16 n_2).$$

On the other hand, it follows from (1) with $n = n_2$ that $|2 - n_2\mu_j| \le 1 - \epsilon_k$, and so $n_2\mu_j \ge 1$. Therefore $\lambda(G_1(j)) \le \mu_j/16 < \mu_j$.

For fixed $j \in \mathbf{Z}_k$, we define a convex subset $G_2(j)$ of $\mathbf{U}^2$ as follows. If $0 < \mu_j \le \frac{1}{2}$, let $G_2(j)$ be the convex hull of the triangle $0 \le x < 1$, $0 \le y < 1$, $y < x$, and of the points $\mathbf{x}_i$ appearing in some $F(n, j)$. Then $\lambda(G_2(j)) > \frac{1}{2} \ge \mu_j$. If $\frac{1}{2} < \mu_j < 1$, let m be the largest integer such that $\mu_j > 1 - 1/m$. Then $G_2(j)$ is taken to be the convex hull of the points $\mathbf{x}_i$ appearing in some $F(n, j)$ and of the open quadrilateral with vertices $\mathbf{x}_{[m/2]}$, $(1, 0)$, $(1, 1)$, and $\mathbf{x}^*$, where $\mathbf{x}^* = (x^*, 1)$ is the intersection of the line $y = 1$ and the tangent to the circle $(x-1)^2 + y^2 = 1$ at $\mathbf{x}_{[m/2]}$. The quadrilateral contains the open rectangle with vertices $(x^*, y_{[m/2]})$, $(1, y_{[m/2]})$, $(1, 1)$, and $(x^*, 1)$ of area

$$(1 - x^*)(1 - y_{[m/2]}) = (1 - y_{[m/2]})^2 (1 - x_{[m/2]})^{-1}.$$

Therefore

$$\lambda(G_2(j)) > (1 - y_{[m/2]})^2 > 1 - 2y_{[m/2]} = 1 - 2 \sin t_{[m/2]}$$
$$> 1 - 2t_{[m/2]} \ge 1 - 1/(4[m/2]) \ge 1 - 1/(m+1) \ge \mu_j,$$

since $4[m/2] \geq m + 1$ for $m \geq 2$. Thus in both cases we have $\lambda(G_2(j)) > \mu_j$.

The proof is completed as in [8]. By construction, we have $G_1(j) \subseteq G_2(j)$, and it was established above that $\lambda(G_1(j)) < \mu_j < \lambda(G_2(j))$. There exists a convex subset S_j of U^2 with $G_1(j) \subseteq S_j \subseteq G_2(j)$ and $\lambda(S_j) = \mu_j$. Again by construction, we have

$$G_1(j) \cap \{\mathbf{x}_1, \cdots, \mathbf{x}_n\} = G_2(j) \cap \{\mathbf{x}_1, \cdots, \mathbf{x}_n\} = F(n, j)$$

for all $n \geq 1$; therefore $S_j \cap \{\mathbf{x}_1, \cdots, \mathbf{x}_n\} = F(n, j)$ for all $n \geq 1$. By (1), we get

$$D(n, S_j) = |Z(n, S_j) - n\lambda(S_j)| = |\text{card } F(n, j) - n\mu_j| \leq 1 - \epsilon_k$$

for all $n \geq 1$, and so $E(S_j) \leq 1 - \epsilon_k$. From the definition of the sets $F(n, j)$, it follows that every $\mathbf{x}_n$ lies in at least one S_j. Furthermore, for $i, j \in \mathbf{Z}_k$ with $i \neq j$, the intersection $S_i \cap S_j$ cannot contain any $\mathbf{x}_n$, for otherwise $F(n, i) \cap F(n, j)$ would be nonempty for some $n \geq 1$.

4. **Concluding remarks.** To what extent is Theorem 2 best possible? We add some remarks concerning this question and related matters.

Remark 1. The constant $1 - \epsilon_k$ in Theorem 2 is best possible. For suppose there exists $\delta_k > \epsilon_k$ such that Theorem 2 holds with $E(S_j) \leq 1 - \epsilon_k$ replaced by $E(S_j) \leq 1 - \delta_k$. We note that for every $n \geq 1$ there exists a unique $j_n \in \mathbf{Z}_k$ with $\mathbf{x}_n \in S_{j_n}$. Consider the sequence $j_1, j_2, \cdots$ in $\mathbf{Z}_k$. We have $A(n, j) = Z(n, S_j)$ for all $n \geq 1$ and all $j \in \mathbf{Z}_k$, and so

$$|A(n, j) - n\mu_j| = |Z(n, S_j) - n\lambda(S_j)| \leq 1 - \delta_k.$$

Thus for any choice of numbers μ_j, $j \in \mathbf{Z}_k$, in $[0, 1]$ with $\sum_{j \in \mathbf{Z}_k} \mu_j = 1$, there would exist a sequence in $\mathbf{Z}_k$ satisfying $|A(n, j) - n\mu_j| \leq 1 - \delta_k$ for all $n \geq 1$ and $j \in \mathbf{Z}_k$. However, this contradicts a result of Tijdeman [9].

Remark 2. If $1 \leq k \leq \infty$ and the μ_j, $j \in \mathbf{Z}_k$, are numbers in $[0, 1]$ with $\sum_{j \in \mathbf{Z}_k} \mu_j < 1$, one may introduce the number $\mu_0 = 1 - \sum_{j \in \mathbf{Z}_k} \mu_j$ and apply Theorem 2. One arrives at a result analogous to Theorem 2, with the bounds on the $E(S_j)$ being $1 - 1/(2k)$ for finite k and 1 for $k = \infty$, and with the last condition replaced by the following one: "every point $\mathbf{x}_n$ of the sequence lies in at most one S_j".

Remark 3. If $2 \leq k \leq \infty$ and the μ_j, $j \in \mathbf{Z}_k$, are numbers in $[0, 1]$ with $\sum_{j \in \mathbf{Z}_k} \mu_j > 1$, then a result analogous to Theorem 2 cannot hold, even if we allow the dependence of the sequence on the μ_j and relax the conditions on the S_j. For suppose $\mathbf{x}_1, \mathbf{x}_2, \cdots$ is a sequence in U^s for which there exist measurable subsets S_j, $j \in \mathbf{Z}_k$, of U^s with $\lambda(S_j) = \mu_j$ and $E(S_j) \leq$

B_k for all $j \in \mathbf{Z}_k$ and some finite B_k, and such that every $\mathbf{x}_n$ lies in a unique S_j. Using the same arguments as in Remark 1, we arrive at a sequence in $\mathbf{Z}_k$ satisfying $|A(n, j) - n\mu_j| \leq B_k$ for all $n \geq 1$ and all $j \in \mathbf{Z}_k$. It follows that $\lim_{n\to\infty} A(n, j)/n = \mu_j$. On the other hand, by choosing $p \in \mathbf{Z}_k$ with $\sum_{j=1}^{p}\mu_j > 1$, we get

$$1 \geq \lim_{n\to\infty} \sum_{j=1}^{p} \frac{A(n, j)}{n} = \sum_{j=1}^{p}\mu_j > 1,$$

a contradiction.

Remark 4. If the sequence $\mathbf{x}_1, \mathbf{x}_2, \cdots$ in Theorem 2 may depend on k and the numbers μ_j, $j \in \mathbf{Z}_k$, a much simpler construction can be given. For each $i \in \mathbf{Z}_k$, define $\lambda_i = \sum_{j=1}^{i}\mu_j$ and $S_i = [\lambda_{i-1}, \lambda_i) \times U^{s-1}$ (with $\lambda_0 = 0$). Choose a point $\mathbf{y}_i \in S_i$. Let $j_1, j_2, \cdots$ be a sequence in $\mathbf{Z}_k$ satisfying the property in Lemma 1_k. Then for the sequence $\mathbf{x}_1, \mathbf{x}_2, \cdots$ in U^s with $\mathbf{x}_n = \mathbf{y}_{j_n}$, $n \geq 1$, the intervals S_i, $i \in \mathbf{Z}_k$, meet all the requirements of Theorem 2.

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