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ON THE CONDITION $c^TA^{-1}b + r > 0$, IN THE LURIE PROBLEM

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ABSTRACT. The problem of Lurie consists in finding NASC's for all solutions of the system $\{x' = Ax + bf(\sigma),\ \sigma' = c^Tx - rf(\sigma)\}$ to tend to zero as $t \to \infty$ under appropriate conditions on the functions involved. When $f(\sigma)/\sigma < M$, for all σ and a certain M, we obtain NASC's for the system to be absolutely stable. When $f(\sigma)/\sigma < M$ as $|\sigma| \to \infty$, we obtain conditions for ultimate uniform boundedness of the solutions of the system.

1. Introduction. We consider a system of real ordinary differential equations

$$x' = Ax + bf(\sigma), \qquad \sigma' = c^Tx - rf(\sigma) \tag{1}$$

where x, b, c are n-vectors, σ and $r > 0$ are scalars, $f(\sigma)$ is a continuous real function such that $\sigma f(\sigma) > 0$ if $\sigma \neq 0$ and A is an $n \times n$ constant matrix with characteristic values which have negative real parts.

The Lurie problem consists in finding NASC's for (1) to be absolutely stable; that is for (1) to be asymtotically stable in the large for any $f(\sigma)$ which satisfies the above conditions.

LaSalle [1] proved that if $r > (Bb + c/2)^TC^{-1}(Bb + c/2)$, then (1) is absolutely stable. Here, B is the unique, symmetric, positive definite solution of the matrix equation $A^TB + BA = -C$, and C is a given symmetric positive definite matrix (see Lefschetz [4, p. 133]). He proved also that the above condition implies

$$c^TA^{-1}b + r > 0, \tag{$*$}$$

and that all solutions are bounded.

Halanay [8, pp. 149, 158] proved that $(*)$ is a necessary condition for

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absolute stability, and that, in the case where b is an eigenvector of A or c an eigenvector of A^T, then (*) is a NASC for absolute stability.

Burton [5] proved that for $f(\sigma)/\sigma \to 0$ as $|\sigma| \to \infty$, then $c^TA^{-1}b + r > 0$ is a NASC for uniform ultimate boundedness of the solutions of (1).

One would like to know if (*) implies uniform, ultimate boundedness or absolute stability for some class of functions $f(\sigma)$.

In the first theorem we prove that if lim sup $f(\sigma)/\sigma < M$ as $|\sigma| \to \infty$ for a certain M, then (*) implies ultimate uniform boundedness of the solutions of (1), and $c^TA^{-1}b + r < 0$ implies the existence of unbounded solutions. We do this by means of a single Liapunov function, thereby greatly simplifying the proof in [5] as well as strengthening the result.

As a corollary we find that (*) is a NASC for absolute stability for the class of functions which satisfy $f(\sigma)/\sigma < M$ for all σ and a certain M. This is equivalent to giving the sector of absolute stability in the sense of Aizermann [6, p. 10].

One then wishes to discover whether or not (*) implies absolute stability for any $f(\sigma)$ such that $\sigma f(\sigma) > 0$. To this end one may note that the work of Lefschetz [2, p. 8] may be used to show that (*) implies absolute stability for $n = 1$.

For the n-dimensional case we observe that (*) would be positive in two cases:

(a) if $r > |c^TA^{-1}b|$, or

(b) $c^TA^{-1}b > 0$.

We prove that (1) is absolutely stable for a condition related to (a) in the corollary to Theorem II, and we give a counterexample to show that (*) does not imply absolute stability for (b).

2. We give here our results on boundedness.

Theorem I. *Let* (1) *be as above. Let* lim sup $f(\sigma)/\sigma < M$ *as* $|\sigma| \to \infty$ *where* $M = (|Bb|\,|c^TA^{-1}| - b^TB(A^{-1})^Tc)^{-1}$. *If* $c^TA^{-1}b + r > 0$, *then the solutions of* (1) *are uniformly ultimately bounded*; *if* $c^TA^{-1}b + r < 0$, *then there are unbounded solutions.*

Corollary I. *If* $f(\sigma)/\sigma < M$ *for all* σ, *then* (*) *is a NASC for absolute stability of* (1).

Corollary II. (*) *is also a necessary condition for absolute stability when we enlarge the class of allowed functions to all the* $f(\sigma)$ *such that* $\sigma f(\sigma) > 0$, $\sigma \neq 0$.

Corollary III. *If the vectors Bb and $(A^{-1})^T c$ are parallel and opposite (i.e., if $(Bb)^T(A^{-1})^T c = -|Bb|\,|(A^{-1})^T c|$), and if $(*)$ holds, then (1) is absolutely stable for all $f(\sigma)$ such that $\sigma f(\sigma) > 0$, $\sigma \neq 0$.*

Proof. Consider the function

$$V = x^T Bx + \alpha(c^T A^{-1}x - \sigma)^2/2$$

with $\alpha > 0$, and $A^T B + BA = -I$. V is positive definite and radially unbounded (i.e., $V(x, \sigma) \to \infty$ as $|(x, \sigma)| \to \infty$). Let us calculate the derivative along the solutions of (1). We have

$$\begin{aligned} V' &= (x^{T\prime}Bx + x^T Bx') + \alpha(c^T A^{-1}x - \sigma)(c^T A^{-1}x' - \sigma') \\ &= (x^T A^T + b^T f(\sigma))Bx + x^T B(Ax + bf(\sigma)) \\ &\quad + \alpha(c^T A^{-1}x - \sigma)[c^T A^{-1}(Ax + bf(\sigma)) - c^T x + rf(\sigma)] \\ &= x^T(A^T B + BA)x + (b^T Bx + x^T Bb)f(\sigma) \\ &\quad + \alpha(c^T A^{-1}x - \sigma)(c^T A^{-1}b + r)f(\sigma). \end{aligned}$$

Calling $k = c^T A^{-1}b + r > 0$, and since $b^T Bx = x^T Bb$ and $c^T A^{-1}x = x^T(A^{-1})^T c$, we get

$$\begin{aligned} V' &= -x^T x + x^T 2Bbf(\sigma) + \alpha k x^T(A^{-1})^T c - \alpha k\sigma f(\sigma) \\ &= -x^T x + x^T(2Bb + \alpha k(A^{-1})^T c)f(\sigma) - \alpha k\sigma f(\sigma). \end{aligned}$$

If we call $2\theta = 2Bb + \alpha k(A^{-1})^T c$, and then complete the square we obtain

$$\begin{aligned} V' &= -x^T x + 2x^T\theta f(\sigma) - \alpha k\sigma f(\sigma) \\ &= -(x - \theta f(\sigma))^T(x - \theta f(\sigma)) + |\theta|^2 f(\sigma)^2 - \alpha k\sigma f(\sigma) \\ &= -(x - \theta f(\sigma))^T(x - \theta f(\sigma)) + |f(\sigma)|(|\theta|^2|f(\sigma)| - \alpha k|\sigma|). \end{aligned}$$

V' would be negative definite if $|\theta|^2\,|f(\sigma)| - \alpha k|\sigma| < 0$, or equivalently if $f(\sigma)/\sigma < M = \alpha k/|\theta|^2$.

From the hypothesis there exists a σ_0 such that $|\sigma| > \sigma_0$ implies $f(\sigma)/\sigma < M$. So for $|\sigma| > \sigma_0$ and all x, V' is negative definite.

We now consider the region $|\sigma| \leq \sigma_0$. The continuous function $|f(\sigma)|$ would reach a maximum N for $|\sigma| \leq \sigma_0$.

We can then write

$$\begin{aligned} V' &= -x^T x + x^T 2\theta f(\sigma) - \alpha k\sigma f(\sigma) \\ &\leq -|x|^2 + |x||2\theta||f(\sigma)| - \alpha k\sigma f(\sigma) \leq -|x|(|x| - |2\theta|N) - \alpha k\sigma f(\sigma), \end{aligned}$$

and V' is negative definite for $|x| > |2\theta|N$.

In this way, for $|x| + |\sigma| > |2\theta|N + \sigma_0 = R$, we have V' negative definite, and that implies uniform ultimate boundedness (see Yoshizawa [7, pp. 38, 42]).

We prove now the second part of the theorem: if $c^TA^{-1}b + r < 0$ $(k < 0)$ then there are unbounded solutions.

Let

$$U = -x^TBx + \alpha(c^TA^{-1}x - \sigma)^2/2,$$

where the letters have the same meaning as in V.

The derivative along the solution is

$$\begin{aligned} U' &= x^Tx - (x^T 2Bb - \alpha k c^T A^{-1}x)f(\sigma) - \alpha k\sigma f(\sigma) \\ &= x^Tx - x^T(2Bb + \alpha|k|(A^{-1})^Tc)f(\sigma) + \alpha|k|\sigma f(\sigma). \end{aligned}$$

We call, as before, $2\theta = 2Bb + \alpha|k|(A^{-1})^Tc$, and we can write $U' = x^Tx - 2x^T\theta f(\sigma) + \alpha|k|\sigma f(\sigma)$.

Observe that $U' = -V'$. Hence, since V' is negative definite outside a certain sphere of radius R, U' is positive definite outside the same sphere.

We claim that this implies the existence of unbounded solutions. To see this, let U_0 be the maximum value of U in the closed sphere of radius $R = 2N|\theta| + \sigma_0$. Pick a point $(0, \sigma_0)$ such that $U(0, \sigma_0) = C_0 > U_0$. U' is positive on the locus $U(x, \sigma) = C_0$; hence, the solution starting at the point $(0, \sigma_0)$ cannot get into the closed sphere $\overline{S}_R$.

Let $R_1 > R$ be an arbitrary number. The solution $[x(t, 0, \sigma_0), \sigma(t, 0, \sigma_0)]$ would leave the sphere $\overline{S}_{R_1}$. For, if we assume not, then the derivative U' would reach a minimum $\delta > 0$ in the compact set $\overline{S}_{R_1} - S_R$. If we integrate along the solution we get

$$U(x(t), \sigma(t)) > U(0, \sigma_0) + \int_0^t \delta\, dt = U(0, \sigma_0) + \delta t,$$

and so $U \to \infty$ on S_{R_1} which is a contradiction.

Proof of Corollary I. If $f(\sigma)/\sigma < M$ for all σ, then (*) implies V' negative definite, and that implies absolute stability, so the condition is sufficient.

To see that it is necessary, we observe that if $c^TA^{-1}b + r = 0$, then the origin is not the only critical point (see Lefschetz [2, p. 19]) and the system cannot be absolutely stable. If $c^TA^{-1}b + r < 0$, then U' is positive definite and we apply Liapunov's theorem on instability (see LaSalle-Lefschetz [3, p. 38]).

Proof of Corollary II. The system is absolutely stable if for all $f(\sigma)$ such that $\sigma f(\sigma) > 0$, $\sigma \neq 0$, the zero solution is asymptotically stable in the large. A subclass of these functions is the class of functions such that $f(\sigma)/\sigma < M$ for all σ. If for this subclass there are unbounded solutions, then the system cannot be absolutely stable, since for some of the allowed functions, there are solutions which do not tend to zero.

Proof of Corollary III. From Corollary I we know that if $f(\sigma)/\sigma < M$, then $(*)$ implies absolute stability. If we can make $M = \infty$, then the sector of stability becomes the entire first quadrant, and any $f(\sigma)$, such that $\sigma f(\sigma) > 0$, satisfies the condition.

If $b^T B(A^{-1})^T c = -|Bb|\,|c^T A^{-1}|$, then $M = \infty$, and that proves the corollary.

We determine now the optimum values of α and M. We have $2\theta = 2Bb + \alpha k(A^{-1})^T c$ and $M = \alpha k/|\theta|^2$. We want to pick α so as to maximize the value of $M = 4\alpha k/|2\theta|^2$.

Calling $y = \alpha k$, we calculate first the value of $|2\theta|^2$.

$$
\begin{aligned}
|2\theta|^2 &= (2Bb + y(A^{-1})^T c)^T (2Bb + y(A^{-1})^T c) \\
&= |2Bb|^2 + y((A^{-1})^T c)^T 2Bb + y(2Bb)^T (A^{-1})^T c + y^2 |(A^{-1})^T c|^2 \\
&= |2Bb|^2 + 2b^T 2B(A^{-1})^T c y + |(A^{-1})^T c|^2 y^2 \\
&= \lambda y^2 + 2\mu y + \nu,
\end{aligned}
$$

where $\lambda = |(A^{-1})^T c|^2$, $\mu = b^T 2B(A^{-1})^T c$, and $\nu = |2Bb|^2$. We can write $M = 4y/(\lambda y^2 + 2\mu y + \nu)$. Then from $M' = 0$ we get $\lambda y^2 + 2\mu y + \nu - y(2\lambda y + 2\mu) = 0$ or $-\lambda y^2 + \nu = 0$, and since $\alpha > 0$ and $k > 0$, we choose $y = +(\nu/\lambda)^{1/2} = |2Bb|/|(A^{-1})^T c|$. We get

$$
\begin{aligned}
M &= 4(|2Bb|/|c^T A^{-1}|)/[|c^T A^{-1}|^2 |2Bb|^2/|c^T A^{-1}|^2 \\
&\qquad + 2b^T B(A^{-1})^T c(|2Bb|/|c^T A^{-1}|) + |2Bb|^2] \\
&= 4|2Bb|/(|2Bb|^2 |c^T A^{-1}| + 2b^T 2B(A^{-1})^T c|2Bb| + |2Bb|^2 |c^T A^{-1}|) \\
&= 4|2Bb|/(2|2Bb|^2 |c^T A^{-1}| + 2b^T 2B(A^{-1})^T c|2Bb|) \\
&= 4/(2|2Bb||c^T A^{-1}| + 2b^T 2B(A^{-1})^T c) \\
&= (|Bb||c^T A^{-1}| + b^T B(A^{-1})^T c)^{-1}
\end{aligned}
$$

which is the maximum value.

3. We know already that $c^TA^{-1}b+r>0$ is a necessary condition for absolute stability, and a sufficient condition for a certain class of functions $f(\sigma)$. We would like to know if this is true in general.

As indicated in the introduction, $c^TA^{-1}b+r$ can be larger than zero if $r>|c^TA^{-1}b|$ or $c^TA^{-1}b>0$.

In the next theorem we find that for $r>|c|\,|2Bb|$, the system is absolutely stable. In case the matrix A is symmetric, we have $2B=-A^{-1}$, and this condition becomes $r>|c|\,|A^{-1}b|$ which implies $r>|c^TA^{-1}b|$.

Theorem II. *Let* (1) *be as above,* $A^TB+BA=-1$, *and let* (*) *hold. If, in addition,* $r(r-2c^TBb)>|c|^2|Bb|^2-(c^TBb)^2$, *then the system is absolutely stable.*

Corollary I. *If* (*) *holds and* $r>|c|\,|2Bb|$, *then* (1) *is absolutely stable.*

Corollary II. *If* (*) *holds and the vectors* Bb *and* c *are parallel and opposite* (*i.e.,* $c^TBb=-|c|\,|Bb|$), *then the system is absolutely stable.*

Proof. Let

$$V=x^TBx+\alpha\int_0^\sigma f(\sigma)\,ds,\qquad \alpha>0.$$

The function V is positive definite. The derivative along the solutions of (1) is

$$V'=-x^Tx+2(Bb+\alpha c/2)^Txf(\sigma)-\alpha rf(\sigma)^2.$$

The condition for V' to be negative definite is $\alpha r-(Bb+\alpha c/2)^T\cdot I^{-1}(Bb+\alpha c/2)>0$ (see Lefschetz [2, p. 132]), or $(Bb+\alpha c/2)^T(Bb+\alpha c/2)-\alpha r<0$, which can be written as

$$|Bb|^2+(Bb)^T(\alpha c/2)+(\alpha c/2)^TBb+(\alpha|c|/2)^2-\alpha r<0, \tag{2}$$

and this in turn as

$$\alpha^2(|c|/2)^2+(c^TBb-r)\alpha+|Bb|^2<0.$$

We try to find an α to satisy this condition. Consider the equation

$$\alpha^2(|c|/2)^2+(c^TBb-r)\alpha+|Bb|^2=0.$$

If it had two real positive roots α_1, α_2, we could pick an α, $\alpha_1\le\alpha\le\alpha_2$, and for this choice V' would be negative definite.

The conditions for this equation to have two positive real roots are

(1) $c^TBb-r<0$ or $r>c^TBb$, and

(2) the discriminant has to be positive. That is $(c^TBb-r)^2$ –

$4(|c|/2)^2|Bb|^2 > 0$, which is true iff

$$(c^TBb)^2 - 2rc^TBb + r^2 - |Bb|^2|c|^2 > 0$$

$$\Leftrightarrow r^2 - 2rc^TBb > |Bb|^2|c|^2 - (c^TBb)^2$$

$$\Leftrightarrow r(r - 2c^TBb) > |Bb|^2|c|^2 - (c^TBb)^2.$$

Since $|Bb|\,|c| > c^TBb$, condition (2) implies condition (1). Also, since V is positive definite, V' is negative definite and $(*)$ holds, it follows that all solutions are bounded and the system is absolutely stable (see LaSalle [1]).

Proof of the corollaries. Note that the second term, $|c|^2|Bb|^2 - (c^TBb)^2$, depends on the orientation of the vectors Bb and c. Several cases may occur:

(a) $c^TBb = |c|\,|Bb|$. The vectors are parallel and in the same direction. The condition becomes then

$$r(r - 2|c||Bb|) > |c|^2|Bb|^2 - |c|^2|Bb|^2 = 0$$

or $r > |c|\,|2Bb|$.

(b) $c^TBb = 0$. The vectors are perpendicular. The condition becomes

$$r(r - 0) > |c|^2|Bb|^2 - 0,$$

or $r^2 > |c|^2|Bb|^2$ from which $r > |c|\,|Bb|$.

(c) $c^TBb = -|c|\,|Bb|$. The vectors are parallel and opposite. Substituting in the condition, we get

$$r(r + |c||2Bb|) > |c|^2|Bb|^2 - (-|c||Bb|)^2 = 0,$$

and so since $r + |c|\,|2Bb| > 0$ is always true, $(*)$ is enough to secure absolute stability.

Observe in the expression of V' that in this case, since Bb and c are parallel and opposite, we can always find an $\alpha > 0$ such that the middle term $Bb + \alpha c/2 = 0$.

That proves Corollary II.

To prove Corollary I, we observe that condition (2) is also true if

$$|Bb|^2 + |Bb||\alpha c/2| + |\alpha c/2||Bb| + (\alpha|c|/2)^2 - \alpha r < 0$$

or

$$\alpha^2|c|^2/4 + \alpha(|Bb||c| - r) + |Bb|^2 < 0.$$

Using the same argument as in the theorem to make this expression negative definite, we arrive at the two following conditions:

(1) $|Bb|\,|c|-r<0$ or $r>|Bb|\,|c|$, and

(2) the discriminant has to be positive, that is $(|Bb|\,|c|-r)^2-4|c|^2/4|Bb|^2>0$, which gives us $r>|2Bb|\,|c|$, and this proves Corollary I.

Remark. In order to compare $r>|c|\,|2Bb|$ with LaSalle's result $r>(Bb+c/2)^TC^{-1}(Bb+c/2)$, we are going to take $C=I$ and use a one-dimensional system.

For the one-dimensional system, the condition $(*)$ is enough to secure absolute stability and is stronger than the other two. Nevertheless we use the one-dimensional case as an illustration to visualize in the parameter space the regions of convergence for the different conditions.

Consider the system

$$x'=-ax+bf(\sigma),\qquad \sigma'=cx-rf(\sigma).$$

Where all the letters are scalars, $a>0$, $r>0$, $A=-a$, $A^{-1}=-a^{-1}$, $2B=a^{-1}$. The condition $r>(Bb+c/2)^T(Bb+c/2)$ becomes $r>(b/2a+c/2)^2=(b/a+c)^2/4$, while $r>|c|\,|2Bb|$ becomes $r>|c|\,|b/a|$, and $r+c^TA^{-1}b>0$ becomes $r>c(b/a)$.

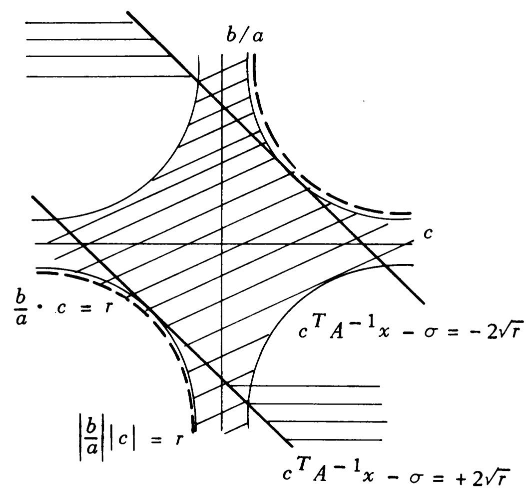

Figure 1

If we plot in the parameter space b/a and c for a fixed r, then: The first condition is represented by the region between the two lines $b/a+c=\pm 2\sqrt{r}$; the second by the region between the four branches of the two hyperbolae $|(b/a)||c|=r$; the third by the region between the two branches of the simple hyperbola $(b/a)c=r$ (see Figure 1).

4. We give a counterexample to show that in the case $c^T A^{-1} b > 0$, the condition (*) does not guarantee absolute stability.

Let

$$x' = -x + \sigma, \qquad y' = -0.01y - \sigma, \qquad \sigma' = 10x + y - \sigma.$$

In this case $f(\sigma) = \sigma$, $c^T = (10, 1)$, $b^T = (1, -1)$, $r = 1 > 0$.

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -0.01 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -100 \end{pmatrix}, \qquad Q = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -0.01 & -1 \\ 10 & 1 & -1 \end{pmatrix}.$$

We check first $c^T A^{-1} b + r > 0$.

$$(10,1)\begin{pmatrix} -1 & 0 \\ 0 & -100 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 = (10,1)\begin{pmatrix} -1 \\ 100 \end{pmatrix} + 1 = -10 + 100 + 1 = 91 > 0.$$

The characteristic equation $|hI - Q| = 0$,

$$\begin{vmatrix} h+1 & 0 & -1 \\ 0 & h+0.01 & 1 \\ -10 & -1 & h+1 \end{vmatrix} = 0.$$

$$(h+1)[(h+0.01)(h+1)+1] - 10h - 0.1 = 0,$$

$$(h+1)[h^2 + 1.01h + 1.01] - 10h - 0.1 = 0,$$

$$h^3 + 2.01h^2 + 2.02h - 10h + 0.91 = 0,$$

$$h^3 + 2.01h^2 - 7.98h + 0.91 = 0.$$

This equation has a positive root between 1.9 and 2.

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Added in proof. Let $c^T Bb = |c||Bb|\cos\theta$. We can substitute the condition in Theorem II by the condition $r > |Bb||c| + c^T Bb = |c||Bb|(1 + \cos\theta)$, which emphasizes the role of the angle θ, between the vector Bb and c. (Suggestion of K. Langenhop (Carbondale).)

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