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HARDY SPACE EXPECTATION OPERATORS AND REDUCING SUBSPACES

JOSEPH A. BALL

ABSTRACT. In this paper we study the range of the isometry on H^p arising from an inner function which is zero at zero by composition. The range of such an isometry is characterized as a closed subspace $\mathfrak{M}$ of H^p (weak-* closed for $p = \infty$) satisfying the following: (i) the constant function 1 is in $\mathfrak{M}$; (ii) if $f \in \mathfrak{M}$ and $g \in H^\infty \cap \mathfrak{M}$, then $fg \in \mathfrak{M}$; (iii) if $f \in \mathfrak{M}$ has inner-outer factorization $f = \chi \cdot F$, then χ is in $\mathfrak{M}$; (iv) if $\{B_\alpha: \alpha \in \mathfrak{A}\}$ is a collection of inner functions in $\mathfrak{M}$, then the greatest common divisor of $\{B_\alpha: \alpha \in \mathfrak{A}\}$ is also in $\mathfrak{M}$; and (v) if $f \in \mathfrak{M}$, $B \in \mathfrak{M}$, where B is inner and $\overline{B} \cdot f \in H^p$, then $\overline{B} \cdot f \in \mathfrak{M}$. The proof makes use of the fact that there exists a projection onto such a subspace satisfying the axioms of an expectation operator, which for $p = 2$, is simply the orthogonal projection. This characterization is applied to give an equivalent formulation of a conjecture of Nordgren concerning reducing subspaces of analytic Toeplitz operators.

1. **Introduction.** Let L^p be the space of Lebesgue measurable functions on the circle whose pth power is integrable, and let H^p be those elements of L^p whose negative Fourier coefficients vanish. For ϕ an inner function with $\phi(0) = 0$, define the operator C_ϕ on L^p by

$$C_\phi: f(e^{i\theta}) \longrightarrow f(\phi(e^{i\theta})). \tag{1}$$

This operator, studied by Nordgren [7] and Ryff [10] among others, has H^p as an invariant subspace, and under our assumptions on ϕ, is an isometry. Our main result is a characterization of the range of C_ϕ considered as an operator on H^p, $1 \le p \le \infty$. See Hoffman [5] or Sz.-Nagy-Foiaş [12] for relevant definitions. The notation g.c.d. means greatest common divisor.

Theorem 1. *A closed* ($p = \infty$, *weak-** *closed*) *subspace* $\mathfrak{M}$ *of* H^p *is the range of an operator* C_ϕ *for some inner* ϕ *with* $\phi(0) = 0$ *if and only if*

(i) $1 \in \mathfrak{M}$, $\mathfrak{M}$ *contains a nonconstant function*;

(ii) *if* $f \in \mathfrak{M}$ *and* $g \in H^\infty \cap \mathfrak{M}$, *then* $fg \in \mathfrak{M}$;

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(iii) *if* $f \in \mathfrak{M}$ *has inner-outer factorization* $f = \chi \cdot F$, *then* $\chi \in \mathfrak{M}$ *and* $F \in \mathfrak{M}$;

(iv) *if* $\{B_\alpha | \alpha \in \mathfrak{A}\}$ *is a collection of inner functions in* $\mathfrak{M}$, *then* $B =$ *g.c.d.* $\{B_\alpha\}$ *is in* $\mathfrak{M}$;

(v) *if* $f \in \mathfrak{M}$, $B \in \mathfrak{M}$, *where* B *is inner and* $\overline{B}f \in H^p$, *then* $\overline{B} \cdot f \in \mathfrak{M}$.

2. **Expectation operators associated with** ϕ. Let $\mathfrak{B}$ be the smallest σ-algebra of Lebesgue measurable sets with respect to which ϕ is measurable. Given any f in L^p, by the Radon-Nikodym theorem there exists a unique $\mathfrak{B}$-measurable function $(P_\phi f)$ such that $\int_B f\,dm = \int_B (P_\phi f)\,dm$ for all $B \in \mathfrak{B}$. The function $(P_\phi f)$ is the Radon-Nikodym derivative of f with respect to $\mathfrak{B}$. The operator P_ϕ is the conditional expectation operator of probability theory and has the following properties [6]:

(2) $P_\phi(1) = 1$;

(3) $P_\phi(f \cdot P_\phi(g)) = (P_\phi f)(P_\phi g)$;

(4) $\|P_\phi f\|_p \le \|f\|_p$, that is, P_ϕ is a contraction operator on L^p;

(5) P_ϕ is a projection, $P_\phi^2 = P_\phi$;

(6) $\int (P_\phi f) \cdot g\,dm = \int f (P_\phi g)\,dm$ for $f \in L^p$, $g \in L^q$, $1/p + 1/q = 1$;

(7) the range of P_ϕ is $L^p(\mathfrak{B})$, the set of functions in L^p measurable with respect to $\mathfrak{B}$, where P_ϕ is considered as an operator on L^p.

We state the following Lemma adapted from Rota [9] without proof.

Lemma 1. *Let* $\mathfrak{A}$ *be the collection of all functions of the form* $p(\phi, \overline{\phi})$, p *a complex polynomial in two variables. Then, for* $1 \le p < \infty$, $L^p(\mathfrak{B})$ *is the closure in* L^p *of* $\mathfrak{A}$, *and* $L^\infty(\mathfrak{B})$ *is the weak-** *closure in* L^∞ *of* $\mathfrak{A}$.

We now use the special assumptions on ϕ. Since ϕ is inner, $\overline{\phi}^j \phi^k = \phi^{k-j}$, and a polynomial $p(\phi, \overline{\phi})$ in ϕ and $\overline{\phi}$ has the form $\Sigma_{j=-n}^{n} c_j \phi^j$. Using this we can obtain

Lemma 2. P_ϕ *leaves* H^p *invariant,* $1 \le p \le \infty$.

Proof. Consider first $p = 2$. Since ϕ is inner with $\phi(0) = 0$, $\{\phi^j: j$ an integer$\}$ is an orthonormal set in L^2. By Lemma 1, this orthonormal set spans $L^2(\mathfrak{B}) \equiv \operatorname{ran} P_\phi$. By (5) and (6), P_ϕ is the orthogonal projection onto $L^2(\mathfrak{B})$. By the form of the spanning orthonormal set, $P_{H^2} P_\phi = P_\phi P_{H^2}$, and the assertion follows for $p = 2$.

Suppose $f \in H^p$, $1 \le p \le \infty$. Then by (6), for all $g \in H_0^\infty$, $\int (P_\phi f) g\,dm = \int f (P_\phi g)\,dm$. Since $g \in H_0^\infty \subset H_0^2$, $(P_\phi g) \in H_0^2 \cap L^\infty = H_0^\infty$, where the last integral vanishes since $f \in H^p$. Hence $P_\phi f \in H^p$.

Lemma 3. *P_ϕ is a projection of L^p onto $C_\phi(L^p)$, and of H^p onto $C_\phi(H^p)$.*

Proof. For $1 \leq p < \infty$, use Lemma 1, the fact that $\bar{\phi}^j \phi^k = \phi^{k-j}$, and that C_ϕ is an isometry. For $p = \infty$, also use the characterization of sequential weak-* convergence as bounded point-wise convergence.

Notation. For convenience of notation, let us set

$$L^p(\phi) = P_\phi(L^p) \ (= C_\phi(L^p)) \quad \text{and} \quad H^p(\phi) = P_\phi(H^p) \ (= C_\phi(H^p)).$$

The shift operator $S: H^p \to H^p$ defined by $S: f(e^{i\theta}) \to e^{i\theta} f(e^{i\theta})$ is an isometry on H^p; the closed (for $p = \infty$, weak-* closed) invariant subspaces of S have been characterized (for $p = 2$, Beurling [1], for arbitrary p, Srinivasan and Wang [11]) as of the form $B \cdot H^p$, where B is inner (i.e., unimodular almost everywhere). If $p = 2$ and $\mathfrak{N}$ is such an invariant subspace with the property that not all $f \in \mathfrak{N}$ have $f(0) = 0$, a nonzero constant multiple of the associated inner function B is obtained as the orthogonal projection of the constant function 1 onto $\mathfrak{N}$ (Hoffman [5, p. 100]). We use this fact in the proof of the next lemma.

Lemma 4. *Let K be a collection of functions in $H^p(\phi)$, and let the smallest closed (for $p = \infty$, weak-* closed) invariant subspace of S containing K be equal to $B \cdot H^p$, where B is the associated inner function. Then $B \in H^p(\phi)$.*

Proof. First consider the case $p = 2$. Without loss of generality, we can assume that not all elements of K vanish at the origin. Otherwise, let j be the largest integer such that $K \subset \phi^j \cdot H^2(\phi)$. Then $K' = \bar{\phi}^j K \subset H^2(\phi)$ and not all elements of K' vanish at the origin. If the invariant subspace of S generated by K' is $B' \cdot H^2$ where B' is inner and in $H^2(\phi)$, then the invariant subspace generated by K is simply $B \cdot H^2$, where $B = \phi^j B'$ is in $H^2(\phi)$.

Hence, assume not all elements of K vanish at the origin, and let $\mathfrak{N}$ be the invariant subspace of S generated by K. Note that finite sums of elements of the form $f \cdot k$, where $f \in H^\infty$ and $k \in K$, form a dense subset of $\mathfrak{N}$. Under our assumption, $c \cdot B$, c some complex number, $0 < |c| \leq 1$, is the orthogonal projection of the constant function 1 onto $\mathfrak{N}$. Hence, by elementary Hilbert space results,

$$\|1 - c \cdot B\|_2^2 = \inf\left\{\|1 - g\|_2^2 : g = \sum_{i=1}^{N} f_i k_i, f_i \in H^\infty, k_i \in K\right\},$$

and if $\{g_n\}$ is any minimizing sequence, g_n converges to $c \cdot B$ in H^2. Note that, for $f_i \in H^\infty$, $k_i \in K$, using properties (2)–(4) of P_ϕ.

$$\left\| 1 - \sum_{i=1}^{N} P_\phi(f_i) \cdot k_i \right\|_2 = \left\| 1 - P_\phi\left(\sum_{i=1}^{N} f_i \cdot k_i\right)\right\|_2$$

$$= \left\| P_\phi\left(1 - \sum_{i=1}^{N} f_i \cdot k_i\right)\right\|_2 \le \left\| 1 - \sum_{i=1}^{N} f_i \cdot k_i \right\|_2 .$$

Hence, if $\{\Sigma_{i=1}^{N_n} f_{i_n} \cdot k_{i_n}\}$ is a minimizing sequence, $\{\Sigma_{i=1}^{N_n} P_\phi(f_{i_n}) \cdot k_{i_n}\} \subset H^2(\phi)$ is also minimizing, $B = \lim_{n\to\infty} \Sigma_{i=1}^{N_n} P_\phi(f_{i_n}) \cdot k_{i_n}$ is in $H^2(\phi)$.

For $p \neq 2$, $p < \infty$, f in H^p having inner-outer factorization $f = \chi \cdot F$, note that $f \to f' = \chi \cdot F^{p/2}$ maps H^p onto H^2 with $\|f\|_p^p = \|f'\|_2^2$, and an invariant subspace closed in H^p norm is mapped onto an invariant subspace closed in H^2 norm. For $p = \infty$, simply use that the closure of $B \cdot H^\infty$ in H^2-norm is $B \cdot H^2$. In this way the general situation is reduced to the case $p = 2$.

3. **Proof of Theorem 1.** We first show necessity in Theorem 1, that is the subspace $H^p(\phi)$ for ϕ an inner function with $\phi(0) = 0$, satisfies conditions (i)–(v) in the statement of Theorem 1. (i) and (ii) are clear. For (v), note that in $f \in H^p(\phi)$, B is inner and in $H^p(\phi)$ such that $\overline{B}f \in H^p$, then $f = P_\phi(f) = P_\phi(B \cdot \overline{B}f) = B \cdot P_\phi(\overline{B}f)$, showing that $P_\phi(\overline{B}f) = \overline{B}f$. For (iii) and (iv) apply Lemma 4 with $K = \{f\}$ and $K = \{B_\alpha | \alpha \in \mathfrak{A}\}$, respectively.

Conversely, assume a subspace $\mathfrak{M} \subset H^p$ satisfies (i)–(v). Assume first that $p = 2$. Set $\phi = \text{g.c.d. } \{B\colon B \text{ inner}, B(0) = 0, B \in \mathfrak{M}\}$. By (i) and (iii), the indicated set is nonvacuous, and by (iv), $\phi \in \mathfrak{M}$. Let $f \in \mathfrak{M}$. We show $f \in H^2(\phi)$. If $f = f(0)$ (a constant), then $f \in H^2(\phi)$. Otherwise $f - f(0)$ has inner part χ in $\mathfrak{M}$ by (ii) and $\chi(0) = 0$. By definition of ϕ, $\overline{\phi}(f - f(0)) \in H^2$, whence by (v), $\overline{\phi}(f - f(0)) \in \mathfrak{M}$, or $f - f(0) \in \phi \cdot \mathfrak{M}$. A similar inductive argument shows that $f - \Sigma_{j=0}^{n} \langle f, \phi^j \rangle \phi^j \in \phi^{n+1}\mathfrak{M}$ ($\langle \cdot , \cdot \rangle$ denotes the inner product for H^2). Hence

$$f - P_\phi(f) = f - \sum_{j=0}^{\infty} \langle f, \phi^j \rangle \phi^j$$

(the infinite series an H^2-limit) is an element of $\bigcap_{k=0}^{\infty} \phi^k \mathfrak{M}$. Since the associated Toeplitz operator T_ϕ is completely nonunitary, $\bigcap_{k=0}^{\infty} \phi^k \mathfrak{M} = (0)$. Hence $f = P_\phi(f) \in H^2(\phi)$, and $\mathfrak{M} \subset H^2(\phi)$. Since $\phi^j \in \mathfrak{M}$ for $j = 0, 1, 2, \cdots$, and $\mathfrak{M}$ is closed, $\mathfrak{M} = H^2(\phi)$.

If $1 \leq p < \infty$, $f \in H^p$ has inner-outer factorization $f = \chi \cdot F$, then the map $f \to f' = \chi \cdot F^{p/2}$ maps a closed subspace of H^p satisfying (i)–(v) onto a closed subspace of H^2 satisfying (i)–(v). For the case $p = \infty$, note $H^\infty \subset H^2$, so one can conclude as in the case $p = 2$ that $f = P_\phi(f)$ for $f \in \mathfrak{M}$, whence $\mathfrak{M} \subset H^\infty(\phi)$. Since $\mathfrak{M}$ is weak-* closed and contains ϕ^j for $j \geq 0$, $\mathfrak{M} = H^\infty(\phi)$.

4. **Analytic Toeplitz operators.** For F an element of H^∞, the associated analytic Toeplitz operator T_F is defined by

$$(T_F f)(e^{i\theta}) = F(e^{i\theta})f(e^{i\theta}) \quad \text{for } f \in H^2.$$

These operators have been much studied and many of their properties are well known (Brown and Halmos [2]).

If b is an inner function which is not a linear fractional transformation, T_b is a shift operator of multiplicity greater than 1, hence T_b has nontrivial reducing subspaces [4]. Hence if F is a function of such an inner function b, any subspace reducing for T_b is also reducing for T_F, whence T_F has nontrivial reducing subspaces. Nordgren [8] has conjectured that this is the only time T_F has nontrivial reducing subspaces. The main result of this section is

Theorem 2. *The following are equivalent:*

(I) *The Toeplitz operator T_F has a nontrivial reducing subspace if and only if F is a function of an inner function which is not a linear fractional transformation.*

(II) (i) *If $F \in H^\infty$ has inner-outer factorization $F = \chi G$ and $\mathfrak{M} \subset H^2$ reduces T_F, then $\mathfrak{M}$ reduces T_χ and T_G.*

(ii) *If $\{B_\alpha = \alpha \in \mathfrak{A}\}$ is a collection of inner functions, $\mathfrak{M} \subset H^2$ reduces T_{B_α} for all $\alpha \in \mathfrak{A}$, then $\mathfrak{M}$ reduces $B = g.c.d.\ \{B_\alpha = \alpha \in \mathfrak{A}\}$.*

For $F \in H^\infty$, set $\mathfrak{A}_F = \{f \in H^\infty$: for any $\mathfrak{M}$ reducing T_F, $\mathfrak{M}$ reduces $T_f\}$. Then $\mathfrak{A}_F$ is a subalgebra of H^∞, and since weak-* convergence in H^∞ corresponds to weak convergence of analytic Toeplitz operators, $\mathfrak{A}_F$ is weak-* closed.

Lemma 5. *For ϕ an inner function, $\phi(0) = 0$, $\mathfrak{A}_\phi = H^\infty(\phi)$.*

Proof. Since $H^\infty(\phi)$ is the weak-* closure of polynomials in ϕ and weak-* convergence in H^∞ corresponds to weak-operator convergence for the associated Toeplitz operators, one has $H^\infty(\phi) \subset \mathfrak{A}_\phi$. Conversely, if $T_f \in \mathfrak{A}_\phi$, since $H^2(\phi)$ is a reducing subspace for T_ϕ and $1 \in H^\infty(\phi)$, $f = T_f(1) \in H^\infty(\phi)$, hence $\mathfrak{A}_\phi \subset H^\infty(\phi)$.

Lemma 6. (I) *is equivalent to*

(III) *for any* $F \in H^\infty$, *there exists an inner function* ϕ *with* $\phi(0) = 0$ *such that* $\mathfrak{A}_F = H^\infty(\phi)$.

Proof. Assume (III), and suppose $\mathfrak{A}_F = H^\infty(\phi)$ and T_F has a nontrivial reducing subspace. Since $\phi \in \mathfrak{A}_F$, T_ϕ has a nontrivial reducing subspace, whence ϕ is not a linear-fractional transformation. Since $F \in H^\infty(\phi)$, F is a function of ϕ.

Conversely, for $F \in H^\infty$, let $\mathfrak{M}$ be the intersection of all subspaces of the type $H^\infty(\phi)$, ϕ inner with $\phi(0) = 0$, containing F. Then, by Theorem 1, $\mathfrak{M} = H^\infty(b)$, where b is inner, $b(0) = 0$, and $F = C_b(G) = G(b)$ for some $G \in H^\infty$. By the construction it follows that G is not a function of a nontrivial inner function. Hence by (I), T_G has no nontrivial reducing subspaces, so the W^* algebra generated by T_G is all bounded operators on H^2. In particular if S: $f(e^{i\theta}) \longrightarrow e^{i\theta} f(e^{i\theta})$ is the standard shift on H^2, there exists a sequence of polynomials $P_n(T_G^*, T_G)$ in T_G^* and T_G converging weakly to S; since $T_G = G(S)$,

$$S = w\text{-}^*\lim_{n \to \infty} P_n(G(S)^*, G(S)).$$

Since T_b, a completely nonunitary isometry, is unitarily equivalent to a direct sum of copies of S,

$$T_b = w\text{-}^*\lim_{n \to \infty} P_n(G(T_b)^*, G(T_b)), \quad \text{or}$$

$$T_b = w\text{-}^*\lim P_n(T_F^*, T_F),$$

so T_b belongs to the W^*-algebra generated by T_F. Hence T_b has at least the reducing subspaces of T_F. Since $F = f(b)$, T_b cannot have any more, hence $\mathfrak{A}_F = \mathfrak{A}_b = H^\infty(b)$, by Lemma 5.

Proof of Theorem 2. Combine Lemma 6 with Theorem 1.

5. **Concluding remarks.** In a related study [3], J. A. Deddens and Tin Kin Wong have made some progress on formulation (II) of the Nordgren conjecture for some special cases.

In a recent preprint, *Entire Toeplitz operators*, I. N. Baker, J. A. Deddens and J. L. Ullman answer the conjecture in the affirmative if ϕ is an entire function.

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DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061