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STRONG UNIFORM DISTRIBUTIONS AND ERGODIC THEOREMS

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ABSTRACT. Let G and H be locally compact σ-compact abelian groups, $\mathfrak{A}$ a mapping from G to H, and $\{\mu_n\}_{n=1}^{\infty}$ a sequence of measures on G. We define the notions: "$\mathfrak{A}$ is a uniform distribution with respect ot $\{\mu_n\}$" and "$\mathfrak{A}$ is a strong uniform distribution". We give a number of examples of these notions and derive some general individual ergodic theorems for measure-preserving transformations with discrete spectrum.

1. **Introduction.** Let G and H be locally compact σ-compact abelian groups, $\mathfrak{A}$ a mapping from G to H and $\{\mu_n;\ n = 1, 2, \cdots\}$ a sequence of finite measures on the Borel sets of G. Below we define the notion "$\mathfrak{A}$ is a uniform distribution with respect to the sequence $\{\mu_n\}$" and "$\mathfrak{A}$ is a strong uniform distribution." We give conditions under which $\mathfrak{A}$ is a strong uniform distribution and show how these can be applied to obtain rather general individual ergodic theorems for measure-preserving transformations with discrete spectrum.

2. **A convergence theorem.** Let G and H be as above and $\bar{G}$ and $\bar{H}$ their Bohr compactifications. Let $\bar{\mu}$ and $\bar{\nu}$ be the respective normalized Haar measures on $\bar{G}$ and $\bar{H}$. If f is an almost periodic (a.p.) function on H (or G), we shall denote by $\bar{f}$ its continuous extension to $\bar{H}$ (or $\bar{G}$). A measure μ on G together with a measurable mapping $\mathfrak{A}$ of G into H induces a measure ν on H by the formula $\nu(E) = \mu[\mathfrak{A}^{-1}(E)]$ for all Borel sets E of H.

Theorem 1. *Assume that $\mathfrak{A}$ is a mapping from G into H such that the composite function $f \circ \mathfrak{A}$ is a.p. on G whenever f is an a.p. function on H. Then the following assertions are equivalent*: (a) *There exists a sequence of finite measures $\{\mu_n\}_{n=1}^{\infty}$ on G which converges weakly to $\bar{\mu}$ such that the induced sequence $\{\nu_n\}_{n=1}^{\infty}$ on H converges weakly to ν. (Note that any measure on G may be considered to be a measure on $\bar{G}$ and similarly for H.)*

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(b) *Assertion* (a) *holds for every sequence* $\{\mu_n\}_{n=1}^\infty$ *of finite measures on* G *converging weakly to* $\bar{\mu}$.

(c) $\int_{\bar{H}} \bar{f}\, d\bar{\nu} = \int_{\bar{G}} \overline{f \circ \mathfrak{A}}\, d\bar{\mu}$ *for every a.p. function* f *on* H.

Proof. First assume (a). Let $\{\mu_n\}_{n=1}^\infty$ be any sequence of finite measures converging weakly to $\bar{\mu}$ and let $\{\nu_n\}_{n=1}^\infty$ be the corresponding induced measures. If f is an a.p. function of H it follows from the hypothesis that $\int_{\bar{G}} \overline{f \circ \mathfrak{A}}\, d\mu_n \to \int_{\bar{G}} \overline{f \circ \mathfrak{A}}\, d\bar{\mu}$. Now $\int_{\bar{H}} \bar{f}\, d\nu_n = \int_{\bar{G}} \overline{f \circ \mathfrak{A}}\, d\mu_n$ by definition of ν_n. Thus $\int_{\bar{H}} \bar{f}\, d\nu_n$ converges to $\int_{\bar{G}} \overline{f \circ \mathfrak{A}}\, d\bar{\mu}$ for every function $\bar{f} \in C(\bar{H})$. But it then follows from (a) that $\int_{\bar{G}} \overline{f \circ \mathfrak{A}}\, d\bar{\mu} = \int_{\bar{H}} \bar{f}\, d\bar{\nu}$ and (b) holds by the definition of weak convergence. That (b) implies (c) is immediate from the above remarks, and similarly that (c) implies (a).

3. **Uniform distributions.** Assume now that H is compact and denote Haar measure on H by ν. If $\mathfrak{A}$ is a Borel mapping of G to H and $\{\mu_n\}_{n=1}^\infty$ is a sequence of bounded measures on G converging weakly to $\bar{\mu}$, we shall say that $\mathfrak{A}$ is a *uniform distribution* with respect to $\{\mu_n\}$ provided the sequence $\{\nu_n\}$ of induced measures converges weakly to ν on H. If, moreover, $\mathfrak{A}$ is a uniform distribution with respect to *every* sequence $\{\mu_n\}$ of bounded measures converging weakly to $\bar{\mu}$ on G, we shall say that $\mathfrak{A}$ is a *strong uniform distribution*. Theorem 1 says that if $f \circ \mathfrak{A}$ is a.p. on G for every $f \in C(H)$, then these two concepts coincide. Actually a little reflection shows that it is sufficient for $\mathfrak{A}$ to have the property that $f \circ \mathfrak{A}$ differs from an a.p. function on G in such a way that the difference converges to zero except on a set whose closure in $\bar{G}$ has Haar measure 0. In that case there will exist an extension $\overline{f \circ \mathfrak{A}}$ to $\bar{G}$ which is continuous except on a set of Haar measure zero.

Let H be compact and $\hat{H}$ be its discrete dual. We shall write $\langle y, \gamma \rangle$ for the character $\gamma \in \hat{H}$. If ν is a measure on H we write its Fourier-Stieltjes transform as

$$\hat{\nu}(\gamma) = \int_H \langle y, \gamma \rangle\, d\nu(y) \quad \text{for } \gamma \in \hat{H}.$$

The following lemma is undoubtedly known and we shall not give its proof.

Lemma 2. *A bounded sequence* $\{\nu_n\}_{n=1}^\infty$ *of measures on a compact abelian group* H *converges weakly to a finite measure* ν *if and only if* $\lim_{n\to\infty} \hat{\nu}_n(\gamma) = \hat{\nu}(\gamma)$ *for all* $\gamma \in \hat{H}$.

Combining Theorem 1 and Lemma 2 we obtain a Weyl criterion for uniform distributions.

Theorem 3. *A mapping* $\mathfrak{A}$ *of* G *into* H *is a uniform distribution with respect to the sequence* $\{\mu_n\}$ *if and only if* $\lim_{n\to\infty}\int_G(\gamma\circ\mathfrak{A})\,d\mu_n = 0$ *for every character* γ *on* H *not the identity.*

An analogous version holds for strong uniform distributions.

4. **Some examples.** In this section we give some conditions for a mapping to be a strong uniform distribution, and some examples.

Theorem 4. *Let* G *be a locally compact* σ*-compact abelian group,* H *a compact abelian group and* $\mathfrak{A}$ *a continuous homomorphism of* G *onto a dense subgroup of* H*. Then for every* $f \in C(H)$ *the composite function* $f\circ\mathfrak{A}$ *is a.p. on* G *and* $\mathfrak{A}$ *is a strong uniform distribution.*

This is essentially a rephrasing of (26.12) in Hewitt and Ross [2].

If q is a positive integer, let Z_q be the cyclic subgroup $\{\exp(2\pi ik/q),\ 0 \le k \le q-1\}$ of the circle group T. With G as above, define for each $\gamma \in \hat{G}$ the group $\phi(\gamma) = T$ if γ has infinite order, and put $\phi(\gamma) = Z_q$ if γ has order q.

Also define $\mathfrak{A}_\gamma(x) = \langle x, \gamma\rangle \in \phi(\gamma)$. Then clearly Theorem 4 applies and we have

Corollary 5. *For each* $\gamma \in \hat{G}$ *the mapping* $\mathfrak{A}_\gamma$ *is a strong uniform distribution of* G *into* $\phi(\gamma)$.

Next we generalize the classical Kronecker theorem. Let $\mathfrak{A}$ be a continuous homomorphism of G into H. Then $\mathfrak{A}$ induces a natural homomorphism $\hat{\mathfrak{A}}$ from the dual $\hat{H}$ of H into the dual $\hat{G}$ of G via the relation

$$\langle \mathfrak{A}(x), \gamma\rangle = \langle x, \hat{\mathfrak{A}}(\gamma)\rangle \quad \text{for all } x \in G \text{ and } \gamma \in \hat{H}.$$

Let $\mathfrak{A}_j\colon G \to H$ $(j = 1, \cdots, m)$ be homomorphisms of G into H. The product homomorphism $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_m$ is the natural homomorphism of G into the product group $\prod_{j=1}^m H$ given by

$$\mathfrak{A}(x) = (\mathfrak{A}_1(x), \cdots, \mathfrak{A}_m(x)) \in \prod_{j=1}^{m} H.$$

We shall say the homomorphisms $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ of G into H are *independent* if $\gamma_j \in H$, $j = 1, \cdots, m$, and $\sum_{j=1}^m \hat{\mathfrak{A}}_j(\gamma_j) = 0$ (the identity in $\hat{G}$) implies that $(\gamma_1, \cdots, \gamma_m)$ is the identity in the dual $\prod_{j=1}^m \hat{H}$ of $\prod_{j=1}^m H$.

Theorem 6. *Let* $\mathfrak{A}_j\colon G \to H$ $(j = 1, \cdots, m)$ *be independent continuous homomorphisms of a locally compact* σ*-compact abelian group* G *into a compact abelian group* H*. Then the product homomorphism* $\mathfrak{A} = \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_m$ *is a strong uniform distribution of* G *into* $\prod_{j=1}^m H$.

The proof is a straightforward application of Theorem 3.

Let $G = Z$ and $H = R/Z$. Suppose $\lambda_1, \cdots, \lambda_m$ are real numbers independent over the integers. Setting $\mathfrak{A}_j(n) \equiv \lambda_j n \pmod 1$, $n \in Z$ $(j = 1, \cdots, m)$ in Theorem 6, we obtain the classical Kronecker theorem.

Corollary 7. *Under the hypotheses of Theorem 6, each $\mathfrak{A}_j$ is a strong uniform distribution of G into H.*

It should be remarked that it is not difficult to construct mappings which are a uniform distribution with respect to some appropriate sequence of measures, but which are not strong uniform. On the other hand we have been unable to decide whether the mapping $n^2\alpha \pmod 1$ with α irrational is a strong uniform distribution. We hope to return to these questions subsequently.

5. **Individual ergodic theorems.** Let $(\Omega, \mathfrak{A}, \nu)$ be a separable Lebesgue space endowed with a nonatomic probability measure, which for simplicity we take to be the unit interval with Lebesgue measure. Let T be a bimeasurable, measure-preserving transformation mapping Ω onto Ω. In [1] it is shown that if $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of probability measures on Z such that $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to Haar measure on $\bar{Z}$, then for $f \in L_2(\Omega)$ we have $\int_Z f(T^k x)\, d\mu_n(k)$ converges in L_2 to Pf, the projection of f onto the subspace of L_2 invariant with respect to T. In fact the condition that $\{\mu_n\}_{n=1}^{\infty}$ should converge weakly to Haar measure on $\bar{Z}$ is shown in [1] to be both necessary and sufficient for the mean ergodic theorem to hold for all $f \in L_2$. We have been unable to date to generalize this result to the individual ergodic theorem, but we are able to prove the following version.

Theorem 8. *Let T be as above and suppose T has countable discrete spectrum. Then there exists $S \subset \Omega$ with $m(S) = 1$ such that for each $x \in S$, f a continuous function on Ω, and $\{\mu_n\}_{n=1}^{\infty}$ a sequence of probability measures on Z converging weakly to Haar measure on $\bar{Z}$, we have*

$$\int_Z f(T^k x)\, d\mu_n \xrightarrow{n} (Pf)(x),$$

where P is as above.

Theorem 8 is the individual ergodic theorem for transformations with discrete spectrum. To prove the theorem we shall show that there exists a set S with $m(S) = 1$ such that for $x \in S$, if we define the mapping $\mathfrak{A}_x$ on Z by $\mathfrak{A}_x(k) = T^k x$, then $\mathfrak{A}_x$ is a strong uniform distribution. The result will then follow by noting that the classical Birkhoff individual ergodic theorem states that there exists a set S with $m(S) = 1$, such that for $x \in S$ we have $\int_Z f(T^k x)\, d\mu_n(k) \to (Pf)(x)$ in the case when for each n the measure μ_n gives mass $1/n$ to each of the integers $1, 2, \cdots, n$. It is easy to verify that this sequence of measures converges to Haar measure on $\bar{Z}$ (see e.g. [1]).

Now if T has countable discrete spectrum we may assume that T is a rotation on a compact metric group, i.e., we may assume that $Tx = x + x_0$, $x \in \Omega$, $x_0 \in \Omega$ and x_0 fixed, with Ω a compact metric group. Then $T^k x = x + kx_0$, and it is trivial to verify that for $f \in C(\Omega)$ we have $\{f(T^k x)\}_{k=-\infty}^{\infty}$ is an a.p. function on Z. The theorem then follows from Theorem 1.

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