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LOCALIZING PRIME IDEMPOTENT KERNEL FUNCTORS

S. K. SIM[1]

ABSTRACT. In this note, we call a prime idempotent kernel functor a localizing prime if it has the so-called property (T) of Goldman. We generalize a theorem of Heinicke to characterize localizing prime idempotent kernel functors and present an example of a prime idempotent kernel functor on Mod-R, the category of unitary right R-modules, which is not a localizing prime, even though R is a right artinian ring.

We shall in most cases follow the terminology in [1]. All rings are assumed to have a unit element and all modules are unitary right modules. Let σ be an idempotent kernel functor on Mod-R, i.e., a left exact subfunctor of the identity functor on Mod-R such that $\sigma(M/\sigma(M)) = 0$ for all $M \in$ Mod-R. We denote the module of quotients of M with respect to σ by M_σ. To each R-module S, there is an associated idempotent kernel functor τ_S given by the formula $\tau_S(M) = \{m \in M \mid f(m) = 0 \text{ for all } f\colon M \to E\}$, where E is the injective hull of S. In case $S = R/I$ for some two-sided ideal I of R, we write μ_I for τ_S, and M_I for the module of quotients of M with respect to μ_I.

An idempotent kernel functor σ on Mod-R is called a *prime* if $\sigma = \tau_S$, where S is a supporting module for σ, i.e., S is σ-torsion free and S/S' is σ-torsion for each nonzero submodule S' of S. A prime idempotent kernel functor σ on Mod-R is called a *localizing prime* if every R_σ-module is σ-torsion free as R-module. In other words, an idempotent kernel functor σ on Mod-R is a localizing prime if and only if σ is a prime with Goldman's property (T) (see [1, Theorem 4.3]).

In [1] Goldman has shown that if A is a commutative ring, then an idempotent kernel functor σ on Mod-A is a prime if and only if $\sigma = \mu_P$ for some

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prime ideal P of A. Thus every prime idempotent kernel functor on Mod-A is a localizing prime. However, for a noncommutative ring, this is not always true.

From [3, Corollary 3.10], we know that if R is a right noetherian ring, then for each prime ideal P of R, μ_P is a prime. In [2, Theorem 4.3] Heinicke has shown that μ_P is a localizing prime if and only if all irreducible R_P-modules are isomorphic and the socle of the R_P-module $(R/P)_P$ is nonzero. We shall show that this result can be generalized to characterize localizing prime idempotent kernel functors on Mod-R without assuming any chain condition on R.

Lemma. *Let σ be an idempotent kernel functor on* Mod-R *with the property that every R_σ-module is σ-torsion free as an R-module. Then an R_σ-module is irreducible if and only if it is a supporting module for σ when regarded as an R-module.*

The proof of this lemma is straightforward and so will be omitted.

Theorem. *Let σ be an idempotent kernel functor on* Mod-R. *Then the following are equivalent:*

(1) *σ is a localizing prime.*

(2) *All irreducible R_σ-modules are isomorphic and, for each R-module M such that $\sigma = \tau_M$, the R_σ-module M_σ has nonzero socle.*

(3) *All irreducible R_σ-modules are isomorphic, and there exists an R-module M such that $\sigma = \tau_M$ and the R_σ-module M_σ has nonzero socle.*

(4) *All irreducible R_σ-modules are isomorphic and σ-torsion free as R-modules.*

Proof. (1) $\Rightarrow$ (2). Since σ is a prime idempotent kernel functor on Mod-R, all σ-injective supporting modules are isomorphic as R-modules by [1, Theorem 6.4]. It follows from the Lemma that all irreducible R_σ-modules are isomorphic as R_σ-modules.

Let M be an R-module such that $\sigma = \tau_M$. Then M contains a supporting module, say U, for σ. Since U_σ is an irreducible R_σ-submodule of M_σ, the R_σ-module M_σ has nonzero socle.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1). Let M be an irreducible R_σ-module. Then by [4, Lemma 1], the R_σ-injective hull (which is also the R-injective hull) of M is a cogenerator of Mod-R_σ which is σ-torsion free as an R-module. Hence every R_σ-module is σ-torsion free as an R-module and $\sigma = \tau_M$. As M is a

supporting module for σ by the Lemma, σ is a prime idempotent kernel functor.

Remark. If σ is a localizing prime idempotent kernel functor on Mod-R, then all irreducible R_σ-modules are isomorphic, and so the Jacobson radical of R_σ is the unique maximal two-sided ideal.

Finally, we present an example showing that a prime idempotent kernel functor on Mod-R need not be a localizing prime, even if R is right artinian.

Example. Let R be the subring of the complete 3×3 matrix ring over a field F consisting of elements of the form

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & a \end{bmatrix}.$$

Then

$$P = \left\{ \begin{bmatrix} 0 & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{bmatrix} \middle| \, b, c, d, e \in F \right\}$$

and

$$K = \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & e \\ 0 & 0 & a \end{bmatrix} \middle| \, a, b, c, e \in F \right\}$$

are two-sided ideals which are also maximal right ideals of R.

Consider the prime idempotent kernel functor $\sigma = \mu_P$. One can show that K is the smallest right ideal of R for which R/K is σ-torsion and then deduce that $\sigma(R) = 0$ and $R_\sigma = \mathrm{Hom}_R(K, R)$. Now, if σ were a localizing prime, then K would have to be a projective R-module, which is not the case.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO, CANADA

Current address: Departamento de Matemáticas, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela