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ON INVERSE LIMITS OF HOMOTOPY SETS

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ABSTRACT. An elementary proof is given that, under certain conditions on a space F, the homotopy set $[X, F]$ maps bijectively onto the inverse limit of homotopy sets determined by the finite subcomplexes of X. The only other satisfactory proof known requires the Brown representability theorem.

Throughout this note we deal only with based maps and based CW complexes. X and F will be such CW complexes, and $\{X_\alpha\}$ will be the set of finite subcomplexes of X, directed by inclusion. We assume that F is connected[2] and that each homotopy group of F is finite.

Theorem 1. *The natural map*

$$[X, F] \xrightarrow{\pi_X^0} \varprojlim_\alpha [X_\alpha, F]$$

is bijective.

This result is trivial when X has dimension 0 or is a finite complex. Moreover, when X is an increasing union of a sequence $\{X_n\}$ of subcomplexes such that $\pi_{X_n}^0$ is surjective for each n, then an inductive application of the homotopy extension property yields that π_X^0 is surjective. In particular, this gives surjectivity when X is a countable CW complex. The problem is that no such straightforward argument seems to work for uncountable X.

Define $L_F^0 X$ to be the inverse limit set in Theorem 1. The theorem may then be interpreted as saying that L_F^0 is a representable functor. This suggests a connection between Theorem 1 and the Brown representability theorem, and, indeed, Brown's theorem has been used to prove Theo-

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[2] There need be no restriction on the number of components of F; our assumption is merely for convenience.

rem 1 [3], [5, p. 3.18]. The purpose of this note is to present a direct, elementary proof.

Theorem 1 is generally false without some kind of finiteness condition on $\pi_i F$ (e.g., see [4]), but the surjectivity of π_X^0, for all X, can be proved using an algebraic condition on F [1], [2]. The precise conditions for surjectivity are not yet well understood.

Now define

$$L_F^1 X = \varprojlim_{\alpha} [X_\alpha \times I \cup X \times \partial I,\ F],$$

where I is the unit interval. (L_F^n can be defined analogously for $n = 2, 3, \cdots$, but they will not be needed.) We then have the following "Mayer-Vietoris" sequence of based sets

$$[X \times I,\ F] \xrightarrow{\pi_X^1} L_F^1 X \underset{i_0}{\overset{i_1}{\rightrightarrows}} [X,\ F] \xrightarrow{\pi_X^0} L_F^0 X,$$

in which π_X^1 is induced by restriction, as in π_X^0, and i_0 and i_1 are induced by the two natural inclusions of X into $X_\alpha \times I \cup X \times \partial I$. Since $i_0 \circ \pi_X^1 = i_1 \circ \pi_X^1$ is a bijection, π_X^1 is injective.

Lemma 1. *The above sequence is exact at* $[X, F]$.

By this we mean that $\pi_X^0 \circ i_1 = \pi_X^0 \circ i_0$, which we take as obvious, *and* that if $\pi_X^0(a) = \pi_X^0(b)$, then $a = i_0(c)$, $b = i_1(c)$ for some $c \in L_F^1 X$.

Theorem 2. π_X^0 *and* π_X^1 *are surjective.*

Therefore, π_X^1 is bijective. Moreover, since π_X^1 is surjective, it follows that $i_0 = i_1$, which, by Lemma 1, forces π_X^0 to be injective. Thus, Theorem 2 implies Theorem 1.

By the remarks made earlier, we can obtain Theorem 2 for X provided that we can prove it for every skeleton of X. It clearly holds for the 0-skeleton of X. Thus, we need only

Lemma 2. *Let* Y *be an* n*-dimensional CW complex,* $n \geq 1$, *and let* X *be a subcomplex of* Y *containing the* $(n-1)$*-skeleton of* Y. *If* π_X^k *is surjective,* $k = 0, 1$, *then so is* π_Y^k.

The proofs of Lemmas 1 and 2 are based on the following two elementary facts:

Fact A. *An inverse limit of nonempty finite sets is nonempty.*

Fact B. *Let* Y *be a CW complex,* X *a subcomplex of* Y, *and* F *as before. Let* g: $X \to F$ *be any map. If* $Y \setminus X$ *consists of finitely many* *cells, then there are only finitely many homotopy classes* rel X *of extensions* $Y \longrightarrow F$ *of* g.

Let $[Y, F]_g$ denote this set of homotopy classes. Its finiteness is a result of elementary obstruction theory. Fact B applies when X is empty, and it still holds, of course, if "rel X" is deleted. Fact A is a direct consequence of König's lemma on finitely branching trees.

Proof of Lemma 1. For each α, there are maps j_0, j_1: $[X_\alpha \times I \cup X \times \partial I, F] \to [X, F]$ determined, as before, by the two natural inclusions of X. Since $\pi_X^0(a) = \pi_X^0(b)$, there exists, for each α, a c_α satisfying $j_0(c_\alpha) = a$, $j_1(c_\alpha) = b$. Letting J_α be the set of all such c_α, we note that the J_α's, together with restriction maps, form an inverse system. Since $X_\alpha \times I$ is a finite CW complex, Fact B implies that J_α is finite. Fact A then produces a $c \in \varprojlim J_\alpha \subset L_F^1 X$ with the required properties. □

Proof of Lemma 2. The proof for π_X^1 is the same as that for π_X^0, and so for notational simplicity we do only the latter.

Case 1. $Y \setminus X$ *has only finitely many cells.*

We redefine $L_F^0 X$ and $L_F^0 Y$ by passing to cofinal subsets of $\{X_\alpha\}$ and $\{Y_\alpha\}$: Namely, we use only X_α containing the boundaries of all cells in $Y \setminus X$, and we use only Y_α of the form $X_\alpha \cup (Y \setminus X)$. This enables us to define the restriction $L_F^0 Y \longrightarrow L_F^0 X$.

Fix α, and note that the commutative diagram

$$\begin{tikzcd} & {[Y_\alpha, F]} \arrow[dr] & \\ {[Y, F]} \arrow[ur] \arrow[dr] & & {[X_\alpha, F]} \\ & {[X, F]} \arrow[ur] & \end{tikzcd}$$

has, by homotopy extension, the following exactness property: If $\phi_\alpha \in [Y_\alpha, F]$ and $\psi \in [X, F]$ restrict to $\psi_\alpha \in [X_\alpha, F]$, then there exists a $\phi \in [Y, F]$ restricting to both ϕ_α and ψ. Let J_α be the set of all such ϕ. By Fact B, J_α is finite.

If $\{\phi_\alpha\} \in L_F^0 Y$ is arbitrary, $\{\psi_\alpha\} \in L_F^0 X$ obtained from it by restriction, and if $\psi \in [X, F]$ satisfies $\pi_X^0(\psi) = \{\psi_\alpha\}$, which ψ exists by hypothesis, then the collection of all J_α that we obtain as above for these ϕ_α, ψ_α and ψ form a system directed by inclusion. By Fact A, $\bigcap J_\alpha$ is nonempty. Each member ϕ satisfies $\pi_Y^0(\phi) = \{\phi_\alpha\}$.

Case 2. *The general case.* So far we have not used the full strength of Fact B: Namely, that it applies to homotopy classes rel X. We now use this.

We begin as before by choosing $\{\phi_\alpha\} \in L^0_F Y$, restricting to $\{\psi_\alpha\} \in L^0_F X$, and pulling back to a class $\psi \in [X, F]$ represented by some $g\colon X \longrightarrow F$. Let $\mathcal{J}$ index the cells of $Y \setminus X$, and let $\{\sigma\}$ be the collection of its finite subsets, directed by inclusion. Define $Y(\sigma) = X \cup \{e_i \mid i \in \sigma\}$, and let $p_\sigma\colon [Y(\sigma), F]_g \longrightarrow [Y(\sigma), F]$ be the standard projection. In Case 1, we showed that image p_σ contains a nonempty finite set J_σ such that $\pi^0_{Y(\sigma)}(J_\sigma) = \{\phi_\alpha\} \mid Y(\sigma)$. Let $K_\sigma = p_\sigma^{-1}(J_\sigma) \subset [Y(\sigma), F]_g$. Using Fact A and arguing as before, we conclude that there is a class $\{\phi'_\sigma\} \in \varprojlim K_\sigma$. For each singleton σ, let f'_σ represent ϕ'_σ, let $\phi' \in [Y, F]_g$ be given by $\bigcup f'_\sigma$, and let $\phi \in [Y, F]$ be given by the same map. For any σ, $\phi' \mid [Y(\sigma), F]_g = \phi'_\sigma$. Letting Y_α be any finite subcomplex of Y, we choose σ so that $Y(\sigma) \supset Y_\alpha$, and we obtain the desired conclusion that

$$\pi^0_Y(\phi) = \{\phi \mid Y_\alpha\} = \{p_\sigma(\phi'_\sigma) \mid Y_\alpha\} = \{\phi_\alpha\}. \quad \Box$$

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