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A REFORMULATION OF THE RADON-NIKODYM THEOREM

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ABSTRACT. The Radon-Nikodym theorems of Segal and Zaanen are principally concerned with the classification of those measures μ for which any $\lambda \ll \mu$ is given in the form

(i) $$\lambda(A) = \int_A g\,d\mu$$

for all sets A of *finite* μ measure.

This paper is concerned with the characterization of those pairs λ, μ for which the equality (i) holds for *every* measurable set A, and introduces a notion of compatibility that essentially solves this problem. In addition, some applications are made to Radon-Nikodym theorems for regular Borel measures.

1. Introduction. The sharpest known form of the Radon-Nikodym theorem to date is due to Segal and appears in Segal [6] and Zaanen [7]. The purpose of this paper is to answer some basic questions left unasked by Segal and Zaanen and, at the same time, attempt to throw some light upon the Radon-Nikodym theorems for regular Borel measures, and upon the way some of these can be made to follow simply from their abstract counterparts.

Notation. Measures μ and λ will be nonnegative and defined on a sigma algebra, Σ, of subsets of a set S. By a measurable set we shall simply mean a member of Σ. A measurable set A is called *μ-null* if $\mu(A) = 0$. If every measurable subset of A with finite μ measure is μ-null, then we say that A is μ-locally null. The nonnull locally null sets of a measure μ can be "killed" by replacing μ by its corresponding *contracted measure* μ^* defined by $\mu^*(A) = \sup\{\mu(B)\colon B \subset A \text{ and } \mu(B) < \infty\}$. (See Zaanen [7, §10].) Clearly, $\mu = \mu^*$ iff μ has no nonnull locally null sets. If every μ-null set is λ-null, then we say that λ is *absolutely continuous* with respect to μ and write $\lambda \ll \mu$. The relation $d\lambda = g\,d\mu$ is said to hold

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μ-locally (respectively, *λ-locally*) in a set A if we have

$$\lambda(B) = \int_B g\, d\mu \tag{1}$$

for every μ-sigma finite (λ-sigma finite) subset B of A. If (1) holds for *every* measurable subset B of A, then the relation $d\lambda = g\, d\mu$ is said to hold *globally* in A. If A is not mentioned then it will be understood that $A = S$. The minimum of measures μ and λ (in the lattice of nonnegative measures defined on Σ) will be denoted as usual by $\mu \wedge \lambda$. A *cross section* of a measure μ is a family $\{g_A: \mu(A)$ is sigma finite$\}$ where for each A, g_A is a nonnegative measurable function which vanishes μ-almost everywhere outside A, and such that whenever A and B are μ-sigma finite sets, the functions g_A and g_B agree μ-almost everywhere in $A \cap B$.

Theorem 1.1 (The classical Radon-Nikodym theorem). *If $\lambda \ll \mu$ and μ is sigma finite, then there exists a measurable function g such that $d\lambda = g\, d\mu$ globally.*

Note that λ need not be sigma finite (Halmos [1, p. 131]). From this theorem the following general result follows easily (Zaanen [7, Theorem 7.3]).

Theorem 1.2. *If $\lambda \ll \mu$, then there exists a cross section $\{g_A\}$ of μ such that whenever a set A is μ-sigma finite, we have $d\lambda = g_A\, d\mu$ globally in A.*

The work of Segal and Zaanen is largely concerned with the question as to when *every* cross section $\{g_A\}$ of a given measure μ can be pieced together into a single measurable function g which will agree μ-almost everywhere with each function g_A on the corresponding set A. They show that this property of μ (called the *R-N property*) is equivalent to the condition that given any λ satisfying $\lambda \ll \mu$, there exists a measurable function g such that $d\lambda = g\, d\mu$ μ-locally. Among the other equivalent conditions that they give are Segal's well-known criterion of localizability and the condition that $L^1(\mu)^* = L^\infty(\mu)$. In addition, they show that if μ is decomposable into finite measures (Hewitt and Stromberg [3, §§ 19.25 and 19.27], and Zaanen [7, Theorem 7.1]), then μ has the *R-N* property. Of course, every sigma finite measure has the *R-N* property.

However, Segal and Zaanen do not give conditions under which the relation $d\lambda = g\, d\mu$ will be valid *globally*, and it is with this question that we are concerned here.

2. **Compatibility.** If λ has no nonnull locally null sets, then whenever $\lambda(A) = \infty$, A must contain subsets of arbitrarily large finite λ measure, and consequently, if for a given μ, the relation $d\lambda = g\,d\mu$ holds λ-locally, it actually holds globally. However, even if neither μ nor λ has nonnull locally null sets, the relation $d\lambda = g\,d\mu$ can quite easily hold μ-locally without holding globally. We may look, for instance, at the famous "Saks example" in which λ is Lebesgue measure and μ is counting measure on the measurable subsets of $[0, 1]$. In this example we see that $d\lambda = 0\,d\mu$ μ-locally, but that there is no function g such that $d\lambda = g\,d\mu$ globally.

Definition. We say that λ is *compatible* with μ if whenever $0 < \lambda(A) < \infty$, there exists a measurable subset B of A such that $\lambda(B) > 0$ and $\mu(B) < \infty$.

Note that any measure is compatible with a sigma finite measure. The following theorem is clear:

Theorem 2.1. *A necessary condition for the existence of a measurable function g such that $d\lambda = g\,d\mu$ globally is that $\lambda << \mu$ and that λ be compatible with μ.*

The converse of Theorem 2.1 does not hold, as we see in the following three examples:

Example 1. (Cf. Zaanen [7, Example 4] or Royden [4, p. 249, Problem 46].) Let M and N be uncountable sets such that $|M| < |N|$, and let $S = M \times N$. Call a set A measurable if, for every horizontal or vertical line L in $M \times N$, either $L \cap A$ or $L \backslash A$ is countable. For each measurable A, let $\mu(A)$ be the number of horizontal or vertical lines L such that $L\backslash A$ is countable, and define $\lambda(A)$ similarly, counting the horizontal lines only. Then although neither μ nor λ has nonnull locally null sets and λ is compatible with μ and $\lambda << \mu$, there is no function g such that $d\lambda = g\,d\mu$ globally, nor even a g such that $d\lambda = g\,d\mu$ μ-locally. Of course, μ does not have the *R-N* property.

Example 2. Let $\mathbf{R}_d$ be the discrete group of reals and let λ be Haar measure on the locally compact group $\mathbf{R}_d \times \mathbf{R}$. Define μ to be λ plus the counting measure on $\mathbf{R}_d \times \{0\}$. Then λ is compatible with μ and $\lambda << \mu$, and although it is easy to find g such that $d\lambda = g\,d\mu$ μ-locally, there is no g such that $d\lambda = g\,d\mu$ globally.

Example 3. Let μ be Haar measure on $\mathbf{R}_d \times \mathbf{R}$ and let $\lambda = \mu^*$. Then λ is compatible with μ, $\lambda << \mu$, λ has no nonnull locally null sets, and $d\lambda = 1\,d\mu$ μ-locally, but there is no g such that $d\lambda = g\,d\mu$ globally.

If λ is finite, the pathology of the above examples disappears.

Theorem 2.2. *If* $\lambda(S) < \infty$, $\lambda << \mu$, *and* λ *is compatible with* μ, *then there exists a measurable function* g *such that* $d\lambda = g\,d\mu$ *globally.*

Proof. Define $\alpha = \sup\{\lambda(A)\colon \mu(A)$ is sigma finite$\}$, and choose an increasing sequence $\{A_n\}$ of measurable sets such that $\lambda(A_n) \to \alpha$ as $n \to \infty$, and define $A = \bigcup_{n=1}^{\infty} A_n$. Then $\lambda(A) = \alpha$ and A is μ-sigma finite. Clearly $\lambda(S \backslash A) = 0$, for otherwise there would exist a measurable subset B of $S \backslash A$ such that $\lambda(B) \neq 0$ and $\mu(B) < \infty$, and we would have $\lambda(A \cup B) > \alpha$, in spite of the fact that $\mu(A \cup B)$ is sigma finite. Using the classical Radon-Nikodym theorem, choose a measurable function g such that $d\lambda = g\,d\mu$ holds globally inside A, and define $g(x) = 0$ for every x outside A. Clearly, $d\lambda = g\,d\mu$ globally. This completes the proof.

In the Saks example (above), λ fails to be compatible with μ in a rather strong way. In fact, λ is *totally incompatible* with μ, by which we mean that $\lambda(A) \neq 0 \Rightarrow \mu(A) = \infty$. By returning to one of the classical proofs of the Radon-Nikodym theorem, we can improve Theorem 2.2.

Theorem 2.3. *Suppose* λ *is finite. Then* λ *can be uniquely decomposed into the sum of three measures* λ_s, λ_t *and* λ_c, *where* λ_s *is singular to* μ, λ_t *is totally incompatible with* μ, *and* λ_c *is both absolutely continuous and compatible with* μ. *A fortiori there exists* $g \in L^1(\mu)$ *such that* $\lambda(A) = \lambda_s(A) + \lambda_t(A) + \int_A g\,d\mu$, *for every measurable set* A.

Proof. Define $\phi = \lambda + \mu$, and let T be the functional on $L^2(\phi)$ defined by $T(f) = \int_S f\,d\lambda$ for every $f \in L^2(\phi)$. Then since

$$|T(f)|^2 \leq \left|\int_S f\,d\lambda\right|^2 \leq \lambda(S)\int_S |f|^2\,d\lambda \leq \lambda(S)\int_S |f|^2\,d\phi,$$

we see that $\|T\| \leq \sqrt{\lambda(S)}$. Therefore, there exists $h \in L^2(\phi)$ such that $0 \leq h \leq 1$ and $\int_S f\,d\lambda = \int_S fh\,d\phi$ for every $f \in L^2(\phi)$. The proof is now concluded by defining λ_s, λ_t and λ_c to be the respective restrictions of λ to the sets of points x where $h(x)$ is, respectively, one, zero, or satisfies $0 < h(x) < 1$, and defining $g = (h/(1 = h))\chi_C$, where $C = \{x\colon 0 < h(x) < 1\}$.

It follows from these theorems that even if the measure μ fails to have the *R-N* property, we can nevertheless obtain the relation $d\lambda = g\,d\mu$ globally when λ is finite, $\lambda << \mu$ and λ is compatible with μ. Furthermore we obtain

Corollary 2.4. *Let* T *be a bounded linear functional on* $L^1(\mu)$, *and suppose there exists a constant* K *such that* $|T(f)| \leq K\|f\|_\infty$ *for every*

$f \in L^1(\mu)$. *Then there exists* $g \in L^\infty(\mu)$ *such that* $T(f) = \int f g \, d\mu$ *for every* $f \in L^1(\mu)$.

Proof. Note that by $\|f\|_\infty$ we mean the least number α such that $\{x\colon f(x) > \alpha\}$ is μ-*locally* null. We may assume $T \geq 0$. Now for every measurable set A define $\lambda(A) = \sup\{T(\chi_B)\colon B \subseteq A$ and $\mu(B) < \infty\}$. Then the measure λ is finite, $\lambda \ll \mu$ and λ is compatible with μ, and so by Theorem 2.2 there exists g such that $d\lambda = g\, d\mu$ globally, and it is easy to see that the function g has all the required properties.

3. The case λ infinite. We have remarked that if λ is infinite, then Theorem 2.1 has no general converse. Positive results may be obtained, however, by imposing certain other restrictions on μ and λ.

Theorem 3.1. *Suppose neither λ nor μ has nonnull locally null sets, $\lambda \ll \mu$, λ is compatible with μ, and that there exists a measurable function g such that $d\lambda = g\, d\mu$ μ-locally. (The latter condition would hold, for example, if μ had the R-N property.) Then for any such function g we have $d\lambda = g\, d\mu$ globally.*

Proof. We need only show that $d\lambda = g\, d\mu$ λ-locally. Suppose $\lambda(A) < \infty$ and, using Theorem 2.2, choose a function h defined in A such that $d\lambda = h\, d\mu$ globally in A. Clearly h and g agree μ-locally almost everywhere, and therefore μ-almost everywhere, in A. Hence $\lambda(A) = \int_A h\, d\mu = \int_A g\, d\mu$, and the result follows.

Theorem 3.2. *Suppose neither λ nor μ has nonnull locally null sets, $\lambda \ll \mu$, λ is compatible with μ, and that λ has the R-N property. Then there exists a measurable function g such that given any measurable set A, there exists a measurable subset B of A such that $\lambda(A) = \lambda(B) = \int_B g\, d\mu$.*

Proof. For every set A of sigma finite λ measure, choose a measurable function g_A which vanishes outside A and satisfies $d\lambda = g_A\, d\mu$ globally in A (see Theorem 2.2). Note that if A and B are two sets of sigma finite λ measure, then g_A and g_B clearly agree μ-locally almost everywhere, and therefore μ-almost everywhere, in $A \cap B$. Therefore since $\lambda \ll \mu$ we see that g_A and g_B agree λ-almost everywhere in $A \cap B$, and it follows that $\{g_A\colon \lambda(A)$ is sigma finite$\}$ is a cross section of λ, and so there exists a measurable function g such that whenever $\lambda(A)$ is sigma finite, g and g_A agree λ-almost everywhere in A.

Now suppose that $\lambda(A) < \infty$, and define $B = \{x \in A: g_A(x) = g(x)\}$. Then $\lambda(A) = \lambda(B) = \int_B g_A \, d\mu = \int_B g \, d\mu$, and, in particular, $\lambda(A) \leq \int_A g \, d\mu$.

On the other hand, if $\lambda(A) = \infty$, we see that A contains subsets B of arbitrarily large finite λ measure, and we obtain

$$\int_A g \, d\mu \geq \sup\left\{\int_B g \, d\mu : B \subseteq A \text{ and } \lambda(B) < \infty\right\}$$
$$\geq \sup\{\lambda(B) : B \subseteq A \text{ and } \lambda(B) < \infty\} = \infty.$$

This completes the proof.

Roughly speaking, small measures are more likely to have the R-N property than large ones. Therefore the following combination of Theorems 3.1 and 3.2 is of interest.

Theorem 3.3. *Suppose neither λ nor μ has nonnull locally null sets, $\lambda << \mu$, λ is compatible with μ, and $\mu \wedge \lambda$ has the R-N property. Then there exists a measurable function g such that the conclusions of Theorem* 3.2 *hold.*

Proof. The result follows on combining Theorems 3.1 and 3.2, after noticing that $\lambda << \mu \wedge \lambda$, and $\mu \wedge \lambda$ clearly has no nonnull locally null sets.

4. **Regular Borel measures.** By a regular Borel measure on a locally compact space S we mean a measure μ constructed from a nonnegative functional as in Rudin [5, Theorem 2.14], or Hewitt and Ross [2, Chapter 11], or Hewitt and Stromberg [3, §§9 and 10]. μ is really an outer measure defined on all subsets of S, and is a measure on the family Σ_μ of μ-measurable sets which contains all Borel sets. Note that if λ and μ are regular Borel measures, and every μ-null set is λ-locally null (for example, we might have $\lambda << \mu$), then $\Sigma_\mu \subset \Sigma_\lambda$, so we may regard both λ and μ as defined on the same sigma algebra Σ_μ. Furthermore, λ is automatically compatible with μ, for if $0 < \lambda(A) < \infty$, then there is a compact subset K of A such that $\lambda(K) > 0$, and of course $\mu(K) < \infty$. Finally, we note from Hewitt and Stromberg [3, Corollary 19.31] that a regular Borel measure has the R-N property.[1]

From these observations, we obtain immediate proofs of the following three theorems (the first two of which are well known).

[1]In saying that a regular Borel measure has the R-N property, we regard μ as defined on all μ-measurable sets, not only on the Borel sets.

Theorem 4.1 (Hewitt and Stromberg [3, Theorem 19.32]). *Let μ be a regular Borel measure, let λ be any measure defined on Σ_μ and suppose $\lambda << \mu$. Then there is a μ-measurable function g such that $d\lambda = g\,d\mu$ μ-locally.*

Proof. The result follows at once since μ has the *R-N* property.

Theorem 4.1 is essentially the same result as Hewitt and Ross [2, Theorem 12.17]. (It really is!)

Theorem 4.2 (Hewitt and Ross [2, Theorem 14.19]). *Let λ and μ be regular Borel measures, suppose $\lambda << \mu$ and suppose $\lambda(S) < \infty$. Then there is a μ-measurable function g such that $d\lambda = g\,d\mu$ globally.*

Proof. The result follows at once from Theorem 2.2 since λ is compatible with μ.

The following result follows at once from Theorem 3.1.

Theorem 4.3. *If λ and μ are regular Borel measures and neither λ nor μ has nonnull locally null sets and $\lambda << \mu$, there is a μ-measurable function g such that $d\lambda = g\,d\mu$ globally.*

Actually, one can do better than this, but the sharper result must be proved directly:

Theorem 4.4. *Let λ and μ be regular Borel measures and suppose that every μ-null set is λ-locally null. Then there is a μ-measurable function g such that $d\lambda = g\,d\mu$ λ-locally.*

Proof. Choose a family $\mathfrak{F}$ of compact sets for the measure λ as in Hewitt and Ross [2, Theorem 11.39]. Let $N = S \setminus \bigcup \mathfrak{F}$. Then N is λ-locally null. For each $F \in \mathfrak{F}$ the measure λ_F, defined by $\lambda_F(A) = \lambda(A \cap F)$ for all Σ_μ-measurable sets A, is a finite measure compatible with μ, and so by Theorem 2.2 there is a μ-measurable function g_F such that $d\lambda_F = g_F\,d\mu$ globally. We may assume that g_F is zero outside F. Define g as follows: If $x \in N$ then $g(x) = 0$, and if $x \in F \in \mathfrak{F}$ then $g(x) = g_F(x)$. Then g is μ-measurable: (This may be a little surprising, since there is no reason to expect $N \in \Sigma_\mu$.) For if $\alpha < 0$ then $\{x\colon g(x) > \alpha\} = S$, and if $\alpha \geq 0$, then $\{x\colon g(x) > \alpha\}$ is μ-measurable since its intersection with any compact set K is of the form $\bigcup_{n=1}^{\infty} K \cap \{x\colon g_{F_n}(x) > \alpha\} \in \Sigma_\mu$. Clearly $d\lambda = g\,d\mu$ λ-locally.

This gives the ultimate result for regular Borel measures:

Corollary 4.5. *Let* λ *and* μ *be regular Borel measures, suppose* λ *has no nonnull locally null sets and* $\lambda \ll \mu$. *Then there exists a* μ*-measurable function* g *such that* $d\lambda = g\,d\mu$ *globally.*

Remark. Example 2 shows that in Corollary 4.5 we cannot dispense with the assumption that λ has no nonnull locally null sets.

Example 3 shows that in Theorem 4.4 and Corollary 4.5 we cannot dispense with the assumption that λ be regular. It also shows that in Theorem 3.1 we cannot dispense with the assumption that μ have no nonnull locally null sets.

Note however, that in Theorem 4.4 and Corollary 4.5, μ *may* have nonnull locally null sets.

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