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ON TYPE-SETS OF TORSION-FREE ABELIAN GROUPS OF RANK 2

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ABSTRACT. We prove existence of a torsion-free abelian group with a prescribed type-set.

The existence of torsion-free abelian groups with prescribed type-set has been studied by Beaumont and Pierce [1] and Koehler [3], [4]. In this paper, we give an improved version of Theorem 7.10 in [1].

We refer the reader to [2] for the concepts of *characteristic, type*, and so forth. The type-set of a group G is denoted by $T(G)$. As in [1], we say that a torsion-free abelian group G is *completely anisotropic* if $\tau(x) \neq \tau(y)$ whenever x and y are linearly independent elements of G and $\tau(x)$ denotes the type of the characteristic of x in G. We say that a set of characteristics $\chi_0, \chi_1, \cdots$ is *relatively disjoint* if $\chi_i > \chi_0$ for all $i \geq 1$ and if, for all primes p and all $i \neq j$, either $\chi_i(p) = \chi_0(p)$ or $\chi_j(p) = \chi_0(p)$. A set of types $\{\tau_0, \tau_1, \cdots\}$ is relatively disjoint if it can be represented by a relatively disjoint set of characteristics. (This is stronger than merely requiring that $\tau_0 = \tau_i \cap \tau_j$ for $i \neq j$.)

Theorem 1. *Let $T = \{\tau_0, \tau_1, \cdots\}$ be a countably infinite relatively disjoint set of types. Then there exists a completely anisotropic rank two torsion-free abelian group A such that $T(A) = T$.*

Proof. It clearly suffices to prove the case $\tau_0 = T(Z)$, where Z is the group of integers. We may then choose characteristics χ_i representing the τ_i in such a way that for all primes p, $\chi_0(p) = 0$ and for all $i \neq j$, either $\chi_i(p) = 0$ or $\chi_j(p) = 0$. Let $\pi_n = \{p \mid \chi_n(p) > 0\}$ for $n \geq 1$, and let $p_n = \min \pi_n$. Thus $\pi_i \cap \pi_j = \varnothing$ for $i \neq j$. We may suppose that the χ_n have been numbered in such a way as to satisfy the following:

(*) $$\text{For all } n,\ p_n < p_{n+1}.$$

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For each $n \geq 0$, choose a rank one group G_n with type τ_n and let x_n be an element of characteristic χ_n in G_n. Let $G = \bigoplus_{n \geq 0} G_n$.

We now arrange the set of elements with integer coefficients $\{ax_0 + bx_1 | b > 0, (a, b) = 1\}$ into a sequence $\{y_n\}$ in such a way that $y_1 = x_1$ and

(**) If $y_n = a_n x_0 + b_n x_1$, then $\max\{|a_n|, b_n\} \leq n$.

From (*) and (**) we get

(***) For all $p \in \pi_n$, $\max\{|a_n|, b_n\} < p$.

Now let $H = \bigoplus_{n \geq 2} \langle y_n - x_n \rangle$. We will now show that G/H is the desired group A. It is clear that G/H has torsion-free rank 2, since the cosets corresponding to x_0 and x_1 are a maximal linearly independent set. We prove next that H is pure in G (i.e. G/H is torsion free). Let $w = \sum_{j \geq 2} e_j(y_j - x_j)$ be any element of H, where the e_j are integers. Since the y_j are linear combinations of x_0 and x_1, it is clear that if $p \notin \bigcup_{n \geq 2} \pi_n$, then $w \in pG$ only if $p | e_j$ for all j. On the other hand, if $w \in pG$ and $p \in \pi_n$ for some $n \geq 2$, we still have $p | e_j$ for $j \neq n$, since $\pi_j \cap \pi_n = \emptyset$. Writing $y_j = a_j x_0 + b_j x_1$, we see that the component of w in G_1 is $\sum b_j e_j x_1$. Since $h_p(x_1) = 0$ and $p \nmid b_n$ by (***), we also see that $p | e_n$, so again $w \in pH$.

We now denote by $\bar{y}_0$ the coset containing x_0, and by $\bar{y}_n$ the coset containing $y_n = a_n x_0 + b_n x_1$, and show that $\tau(\bar{y}_n) = \tau(x_n)$. It suffices, of course, to show that $\tau(\bar{y}_n) \leq \tau(x_n)$. For $n = 0$ or $n = 1$, this is easy. For if $h_p(\bar{y}_n) \geq s > h_p(x_n)$ for some p, then there must exist c_i such that $h_p(x_n + \sum_{i \geq 2} c_i(y_i - x_i)) \geq s$. Then clearly p^s divides all c_i (except for at most one c_j, if $p \in \pi_j$ for some $j \geq 2$). But then comparing the coefficients of x_1 (in case $n = 0$) or x_0 (in case $n = 1$), using (***) shows us that $p^s | c_j$ as well, and this yields a contradiction.

For $n \geq 2$, we consider two classes of primes.

1. If $p \notin \bigcup_{i \neq n} \pi_i$, we claim that $h_p(\bar{y}_n) = h_p(x_n)$.

Suppose there exist coefficients c_i so that

$$y_n + \sum_{i \geq 2} c_i(y_i - x_i) = (1 + c_n)y_n - c_n x_n + \sum_{i \neq n} c_i(y_i - x_i)$$

has height greater than $h_p(x_n)$. Checking the coefficients of x_i for $i \geq 2$, we see that $p | c_i$ for all i; this fails because, by (***), $h_p(y_n) = 0$.

2. Consider those p such that $p \in \pi_j$ for some $j \neq n$. We claim that $h_p(\bar{y}_n) \neq \infty$ and for almost all such p, $h_p(\bar{y}_n) = 0$. Now

$$(1) \qquad h_p(\overline{y}_n) \leq h_p(b_j\overline{y}_n) = h_p(b_n\overline{y}_j + (a_nb_j - a_jb_n)\overline{x}_0).$$

Since $h_p(\overline{y}_j) \geq h_p(x_j)$ and $h_p(\overline{x}_0) = 0$ and $a_nb_j - a_jb_n \neq 0$, this means that $h_p(\overline{y}_n) \neq \infty$ if $h_p(x_j) = \infty$. On the other hand if $h_p(x_j) < \infty$ and $h_p(\overline{y}_n) = \infty$, then there must exist c_i so that $h_p(y_n + \Sigma_{i\geq 2}c_i(y_i - x_i)) > h_p(x_j)$. But in this case $p|c_i$ for all i and we get a contradiction. Thus $h_p(\overline{y}_n) \neq \infty$.

It remains to show that there are at most a finite number of pairs (j, p) such that $j \neq n$, $p \in \pi_j$, and $h_p(\overline{y}_n) > 0$. Now since $h_p(b_n\overline{y}_j) > 0$ and $h_p(\overline{x}_0) = 0$, we see from (1) that $h_p(\overline{y}_n) = 0$ whenever $p \nmid (a_nb_j - a_jb_n)$. Now let $p_j = \min \pi_j$. By (*) and a well-known theorem in number theory (for example [5, p. 10]), $\lim_{j\to\infty} j/p_j = 0$, so that, by (**), $\lim a_j/p_j = \lim b_j/p_j = 0$. Thus for sufficiently large j, $|a_nb_j - a_jb_n| < p_j \leq p$ for all $p \in \pi_j$. Therefore there are at most finitely many pairs (j, p) so that $p|(a_nb_j - a_jb_n)$. This completes the proof that $\tau(\overline{y}_n) = \tau(x_n)$ for all n. Since every element of G/H is a rational multiple of some $\overline{y}_n$, the theorem is proved.

Considering the proof of Theorem 1, we can easily see that the following theorem also holds.

Theorem 2. *Let $T' = \{\tau_1, \tau_2, \cdots\}$ be a type-set such that τ_0 is excluded from the type-set T of Theorem 1. Then there exists a completely anisotropic torsion-free abelian group A' of rank 2 such that $T(A') = T'$.*

Remark. Let p_n denote the nth prime from the smallest. Let $r_n = p_{2n-1}$, and $s_n = p_{2n}$. Let $\Pi_1 = \{r_n\}$ and $\Pi_2 = \{s_n\}$. Let χ_0, χ_1, λ_i, and μ_j be the characteristics such that:

$$\chi_0(p) = 0 \quad \text{for every prime } p,$$

$$\chi_1(p) = 0 \quad \text{for } p \in \Pi_1, \qquad \chi_1(p) = 1 \quad \text{for } p \in \Pi_2,$$

$$\begin{array}{lll} \lambda_i(r_i) = \infty, & \lambda_i(p) = 0 \quad \text{for } p \neq r_i, & \\ \mu_j(s_j) = \infty, & \mu_j(p) = 0 \quad \text{for } p \neq s_j, & \end{array} \qquad (i, j = 1, 2, 3, \cdots).$$

Let $T = \{[\chi_0], [\chi_1], [\lambda_1], [\lambda_2], \cdots, [\mu_1], [\mu_2], \cdots\}$, where $[\chi]$ denotes the type corresponding to the characteristic χ. Then T is neither relatively disjoint nor satisfies the condition of Theorem 2. But we can construct a completely anisotropic abelian group A of rank 2 such that $T(A) = T$.

The type-set of Example on p. 190 in [3] cannot be realized as type-set of any completely anisotropic group of rank 2.

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