PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 1, March 1975

CLASSIFICATION OF CONTINUOUS FLOWS ON 2-MANIFOLDS

DEAN A. NEUMANN

ABSTRACT. We prove that a continuous flow with isolated critical points on an arbitrary 2-manifold is determined up to topological equivalence by its separatrix configuration.

1. **Introduction.** In [3] Markus proves the following result: If ϕ is a C^1 flow on the plane, with isolated critical points and no limit separatrices other than critical points, then ϕ is determined up to topological equivalence by its separatrix configuration. The purpose of the present paper is to extend this result to continuous flows on arbitrary 2-manifolds and remove the restriction on limit separatrices.

2. **Definitions and preliminaries.** Let M denote a 2-manifold (separable metric, connected and without boundary, but not necessarily compact nor orientable) and $\phi: M \times \mathbf{R}^1 \longrightarrow M$ a continuous flow on M. Two such flows, (M_1, ϕ_1) and (M_2, ϕ_2), are (*topologically*) *equivalent* if there is a homeomorphism of M_1 onto M_2 which takes orbits of ϕ_1 onto orbits of ϕ_2, preserving sense.

We call (M, ϕ) *parallel* if it is equivalent to one of the following:

1. $\mathbf{R}^2$ with flow defined by $y' = 0$;

2. $\mathbf{R}^2 - \{0\}$ with flow defined (in polar coordinates) by $dr/dt = 0$, $d\theta/dt = 1$;

3. $\mathbf{R}^2 - \{0\}$ with flow defined by $dr/dt = r$, $d\theta/dt = 0$;

4. $S^1 \times S^1$ with 'rational' flow (e.g., the flow induced by (1) above, under the usual covering map).

We distinguish these as *strip, annular, spiral* (or *radial*) and *toral* respectively.

Throughout this paper we consider flows (M, ϕ) with isolated critical points. Denote the orbit ($\pm$ semiorbit) of $x \in M$ by $\gamma(x)$ ($\gamma^{\pm}(x)$) and let

$$\alpha(x) = \overline{\gamma^-(x)} - \gamma^-(x), \qquad \omega(x) = \overline{\gamma^+(x)} - \gamma^+(x).$$

Received by the editors December 12, 1973.
AMS (MOS) subject classifications (1970). Primary 34C35; Secondary 34C40.
Key words and phrases. Flows on 2-manifolds, separatrices.

We say that $\gamma(x)$ is a *separatrix* of ϕ (cf. [3]) if $\gamma(x)$ is not contained in a *parallel* neighborhood N satisfying both:

1. for all $y \in N$, $\alpha(y) = \alpha(x)$ and $\omega(y) = \omega(x)$;

2. $\overline{N} - N$ consists of $\alpha(x)$, $\omega(x)$ and exactly two orbits $\gamma(a), \gamma(b)$ of ϕ with $\alpha(a) = \alpha(b) = \alpha(x)$ and $\omega(a) = \omega(b) = \omega(x)$.

Let S denote the union of all separatrices of ϕ—so S is a closed invariant subset of M. A component of the complement, with the restricted flow, is called a *canonical region* of ϕ.

Lemma. *Any canonical region of* (M, ϕ) *is parallel.*

Proof. Let $(R, \phi' = \phi|R)$ be a canonical region. There are no separatrices in R, so the set consisting of orbits homeomorphic with S^1 is open, and similarly for the set consisting of line homeomorphs. Hence R consists entirely of closed orbits or entirely of line orbits.

Also, two orbits of ϕ' can be separated with disjoint parallel neighborhoods. For suppose $\gamma(x)$ and $\gamma(y)$ are distinct orbits (closed or not) which cannot be separated. Then, for any parallel neighborhood N_x of x, we have $y \in \overline{N}_x$; i.e., $y \in \bigcap \overline{N}_x = \alpha(x) \cup \gamma(x) \cup \omega(x)$. But then $y \in \alpha(x)$ (or $y \in \omega(x)$) and this is impossible since y lies in a parallel neighborhood which may be taken to exclude $\gamma(x)$.

It follows that the quotient space R/ϕ' is a (Hausdorff) 1-manifold and hence that the natural projection $\pi\colon R \to R/\phi'$ is a locally trivial fibering of R over $\mathbf{R}^1$ or S^1, with fibers homeomorphic to $\mathbf{R}^1$ or S^1. Since the flow provides a natural orientation on the fibers, there are only four possibilities—the four classes of parallel flows described above.

A *separatrix configuration* for (M, ϕ), denoted S^+, is the union of all separatrices of ϕ together with a representative orbit from each canonical region of ϕ. Separatrix configurations, S_1^+ for (M_1, ϕ_1) and S_2^+ for (M_2, ϕ_2) are *equivalent* if there is a homeomorphism of M_1 onto M_2 taking orbits of (S_1^+, S_1) onto those of (S_2^+, S_2), preserving sense. A separatrix $\gamma(x)$ of ϕ is called a *limit separatrix* if $\gamma(x)$ is in the closure of $S - \gamma(x)$.

It follows from the Lemma that any canonical region R admits a complete transversal; i.e., a section which meets each orbit of R exactly once. We will also use repeatedly the fact that through any nonrest point of (M, ϕ) there is a local section S, with $S[-\epsilon, \epsilon]$ homeomorphic to the rectangle $\{(s, t) | s \in [-1, 1], |t| \leq \epsilon\}$ under the map $(s, t) \to \phi(\alpha(s), t) = \alpha(s) \cdot t$, where $\alpha\colon [-1, 1] \to S$ defines the section S. If x is wandering, we may take $\epsilon = \infty$ (see [1, Chapter IV, § 2], and [2, Theorem 1]).

If R is a canonical region of (M, ϕ), let ∂R denote $\overline{R} - R$. In the simplest situations each noncritical point of ∂R is *accessible* from R as the endpoint of a (local) section of ϕ which otherwise lies in $\dot{R}$. However, there may be limit separatrices in ∂R which are not accessible from R (for example, if we insert a rest point into a spiral region with a limit cycle, we obtain a strip region R, with the limit cycle in ∂R but not accessible from R). Hence we distinguish the union of separatrices accessible from R as the *accessible boundary* of R, and denote it by δR. It is not hard to show that every (noncritical) boundary point of a spiral or annular region R is accessible from R.

Finally, we distinguish two types of spiral regions. Suppose $\gamma(m)$ is an orbit of the spiral region R and that both $\alpha(m)$ and $\omega(m)$ contain noncritical points. We say that

1. R is *orientable* if the orientations on separatrices of δR induced by the flow are compatible with some orientation of R (cf. Figure 1);

2. R is *nonorientable* otherwise.

We say that an arc *spans* a canonical region R, if it is a (local) section which lies in R except for its endpoints. Note that there can be no spanning section in an orientable spiral region. Hence such regions cannot accumulate at a noncritical limit separatrix.

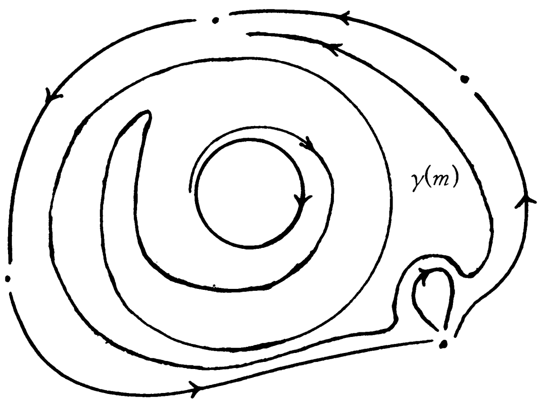

Figure 1

3. Subdivisions of canonical regions. Suppose ϕ_1 and ϕ_2 are continuous flows on M with isolated critical points and the same separatrix configuration S^+. For each canonical region R, we wish to describe an equivalence h of (R, ϕ_1) with (R, ϕ_2) which extends by the identity to an equivalence on $R \cup \delta R$. We do this by constructing subdivisions of (R, ϕ_1), (R, ϕ_2) which 'converge' at δR, and defining h to be 'cellular' in these subdivisions. The construction also restricts h on the interior of R in such a way that the composite equivalence (obtained by piecing together the various canonical regions) extends continuously to the limit separatrices. The latter restriction is measured by a positive constant ϵ, which we assume fixed for the remainder of this section.

It is convenient to pass to the manifold $\check{M}$ consisting of M minus the critical points of ϕ_i; we denote the restricted flows by ϕ_i also. We may assume that the topology of $\check{M}$ is defined by a complete metric ρ. The constructions of this section refer to $(\check{M}, \phi_i)$.

Strip canonical regions. Let R be a strip region and let $\gamma(m) \subset S^+$ be the distinguished orbit. Choose points $p_k \in \gamma(m)$ $(k \in \mathbf{Z})$ satisfying:

1. $p_k = mt_k$ where t_k strictly increases with k and is unbounded above and below;

2. $\rho(p_k, p_{k+1}) < \epsilon$ $(k \in \mathbf{Z})$;

3. if $\alpha(m) \neq \emptyset$ $(\omega(m) \neq \emptyset)$ then $\lim_{k\to-\infty}\rho(p_k, p_{k+1}) = 0$ $(\lim_{k\to\infty}\rho(p_k, p_{k+1}) = 0)$.

Note that $\gamma(m)$ separates R into two half-regions R^+ and R^- (both containing $\gamma(m)$). If $\delta R \neq \emptyset$, we construct a subdivision of R^+; R^- is treated similarly.

Define

$$a_k = \inf\{a > 0 \mid \exists \text{ a section of } \phi_1 \text{ from } p_k \text{ to } \delta R^+ \text{ of diameter } a\}.$$

Let $A \subset \mathbf{Z}$ consist of 0 and those indices k for which $a_k \leq 1$. Construct disjoint sections S_k $(k \in A)$ of ϕ_1 from p_k to points $q_k \in \delta R^+$, with $\operatorname{diam}(S_k) < 2a_k$. We may see that this is possible as follows. If we have already constructed n such sections we can add another, possibly having to adjust some of the previously constructed ones to insure disjointness. However, the section at a given p_{k_0} need be adjusted only a finite number of times in this process. If $\omega(m) = \emptyset$ $(\alpha(m) = \emptyset)$, this follows from the fact that $\rho(p_{k_0}, p_k) \to \infty$ as $k \to \infty$ $(k \to -\infty)$ (because ρ is complete), while the sections constructed have bounded diameters. If $\omega(m) \neq \emptyset$ $(\alpha(m) \neq \emptyset)$, then there are indices for which a_k is arbitrarily small; once S_{k_0} is adjusted to miss a sufficiently small section, subsequent sections may be chosen disjoint from S_{k_0} without altering it.

Next, for $k \in A$, let

$$b_k = \inf\{b > 0 \mid \exists \text{ a section of } \phi_2 \text{ from } p_k \text{ to } q_k \text{ of diameter } b\},$$

and construct disjoint sections S'_k of ϕ_2 from p_k to q_k with $\operatorname{diam}(S'_k) < 2b_k$.

Finally, let $\{d_k\}_{k\in \mathbf{Z}^+}$ be a countable dense subset of the separatrices of δR^+ which is disjoint from $\{q_k\}_{k\in A}$. Construct disjoint sections T_k of

ϕ_1 and T'_k of ϕ_2, both terminating at d_k and satisfying (cf. Figure 2):

1. T_k is disjoint from every S_k (T'_k is disjoint from every S'_k);
2. $\operatorname{diam}(T_k) \to 0$ ($\operatorname{diam}(T'_k) \to 0$) as $k \to \infty$;
3. if T_k (T'_k) has initial point on the orbit $\gamma_1(r_k)$ of ϕ_1 ($\gamma_2(r'_k)$ of ϕ_2), where $r_k \in S_0$ ($r'_k \in S'_0$), then r_k (r'_k) converges *monotonically* to q_0.

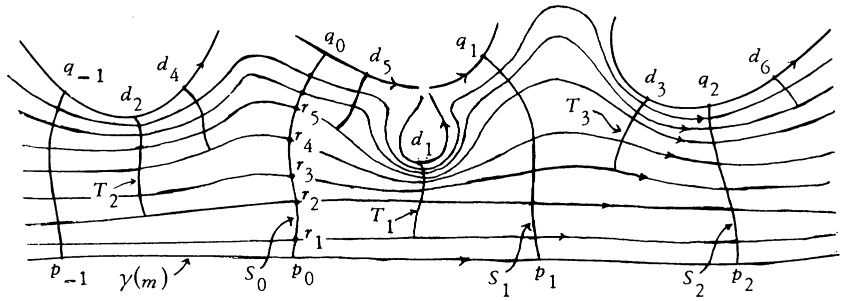

Figure 2

The sections S_k, T_k and the orbits $\gamma_1(r_k)$ then partition R^+ into a locally-finite collection of 2-cells (those at the 'ends' of R^+ missing a closed subarc of their boundaries), which we refer to as an *ϵ-subdivision* of R^+ with respect to ϕ_1. The S'_k, T'_k and $\gamma_2(r'_k)$ provide an 'isomorphic' subdivision of R^+ with respect to ϕ_2. It follows that there is an equivalence h of (R^+, ϕ_1) onto (R^+, ϕ_2) which takes cells of one subdivision onto the corresponding cells of the other. If we define h to be the identity on δR^+ then the extended function is continuous at δR^+. If $p \in \delta R^+$ is not in $\alpha(m) \cup \omega(m)$, this follows easily from the construction: let U be an arbitrary neighborhood of p in $R^+ \cup \delta R^+$; pick i, j and l so that the neighborhoods N (N') of p bounded by segments of T_i, T_j T_j and $\gamma_1(r_l)$ (T'_i, T'_j and $\gamma_2(r'_l)$) both lie in U; then $h(N) = N' \subset U$. If $p \in \alpha(m) \cup \omega(m)$, then $\gamma(p)$ is a limit separatrix and continuity of h at p follows by the general argument for limit separatrices given in § 4.

If $\delta R^+ = \emptyset$ then $\partial R = \emptyset$ and we may take h to be any equivalence of (R, ϕ_1) with (R, ϕ_2) which is the identity on $\gamma(m)$.

Annular canonical regions. Here the construction is exactly as above, except that $\{p_k\}$ is now a finite sequence, spaced less than ϵ apart and monotonic along the distinguished orbit $\gamma(m)$.

Spiral canonical regions. First suppose that R is a nonorientable spiral region. Let S_0 be a local section of ϕ_1 which spans R and has diameter less than twice the infimum for such sections. Let p and q be two successive intersections of $\gamma^+(m)$ with S_0 and pick $p_0 = p, p_1, \cdots, p_n, p_{n+1} = q$ monotonic along $\gamma^+(p_0)$ and spaced closer together than ϵ. Let C denote the simple closed curve consisting of $[p_0, p_{n+1}] \subset \gamma^+(m)$ and $[p_0, p_{n+1}] \subset S_0$, and define $R^+ = C[0, \infty)$, $R^- = C(-\infty, 0]$.

Construct disjoint sections S_k of ϕ_1 from p_k to points $q_k \in \delta R^+$ of diameter less than twice the infimum of possible diameters. Then construct sections S'_k to ϕ_2 from p_k to q_k with the analogous restriction on diameters.

If R is orientable (or if $\alpha(m)$ or $\omega(m)$ is empty), then we need not construct such spanning sections. However, we may define the analogues of C, R^+ and R^- in these cases also.

Finally, for any spiral region R with $\delta R^+ \neq \emptyset$, let $\{d_k\}$ $(k \geq 1)$ be a countable dense subset of δR^+ (disjoint from $\{q_k\}$ in the nonorientable case) and construct local sections T_k (T'_k) to ϕ_1 (ϕ_2 respectively) satisfying (cf. Figure 3):

1. T_k and T'_k originate at the same point of $\gamma(m)$;
2. T_k and T'_k terminate at d_k;
3. $\text{diam}(T_k) \to 0$ and $\text{diam}(T'_k) \to 0$.

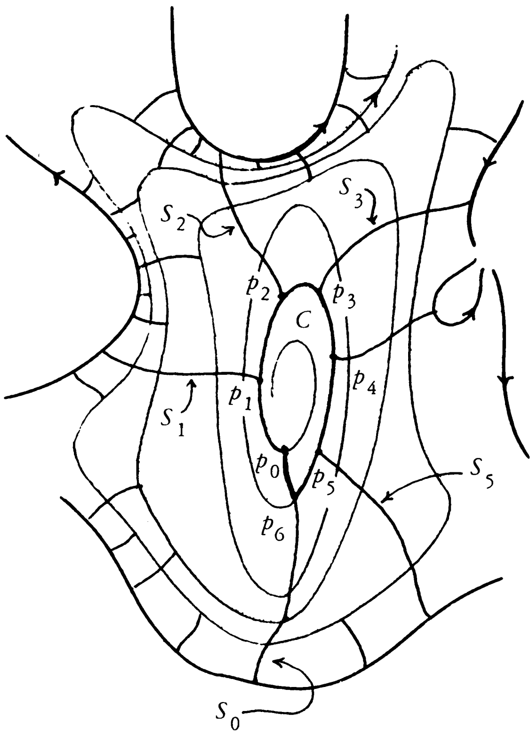

Figure 3

There is an equivalence h of (R^+, ϕ_1) onto (R^+, ϕ_2) which is the identity on C, and takes cells of one partition onto corresponding cells of the other. As in the case of strip regions, such an equivalence extends continuously to δR^+ by the identity.

If $\delta R^+ = \emptyset$, take h to be any equivalence which is the identity on C. R^- is treated similarly.

Toral canonical regions. If R is a toral region then $R = M$. Let h be any equivalence of (M, ϕ_1) with (M, ϕ_2).

4. Classification theorem.

Theorem. *Suppose ϕ_1 and ϕ_2 are continuous flows on the 2-manifold M, with isolated critical points. Then ϕ_1 and ϕ_2 are equivalent iff their separatrix configurations are equivalent.*

Proof. (*Sufficiency*). Suppose k is an equivalence of (S_1^+, S_1) with (S_2^+, S_2). If h is a homeomorphism of M which is the identity on S_2, and an equivalence of the flow induced by ϕ_1 under k with ϕ_2, then hk is the required equivalence. Hence we may assume that ϕ_1 and ϕ_2 have the same separatrix configuration S^+, and construct h.

Order the canonical regions $\{R_n\}$ $n \geq 1$. Let $\gamma(m_n)$ denote the distinguished orbit of R_n. For each n, define $1/n$-subdivisions of R_n with respect to ϕ_1 and ϕ_2, as above. By the results of § 3, there is a cellular equivalence of (R_n, ϕ_1) with (R_n, ϕ_2), which extends by the identity to nonlimit separatrices of δR_n. Define h to be the identity on S^+; we need to prove that h is continuous at limit separatrices.

First suppose p is a noncritical point on a limit separatrix and fix $\epsilon > 0$. Then $\gamma(p)$ separates a neighborhood U of p into two components; at least one of these, say H, meets separatrices which accumulate at p. Let N denote a closed trivial neighborhood of p in $\overline{H}$ which is bounded by local sections of ϕ_1 terminating on $\gamma(p)$, and a segment of a separatrix $\gamma(q)$ (cf. Figure 4), and let $N' \subset N$ be similarly bounded by $\gamma(q)$ and sections of ϕ_2. Let h_1 (h_2') be a homeomorphism of N (N') onto $D = \{(x, y) | \, |x| \leq 1, 0 \leq y \leq 1\}$, taking orbit segments of ϕ_1 (ϕ_2) onto horizontal segments and taking p to 0. Let $k: D \to D$ be an embedding which extends the map $h_1 \circ h_2'^{-1}|_{h_2'(N' \cap S^+)}$, and maps horizontal segments to horizontal segments. Define $h_2: N' \to D$ by $h_2 = kh_2'$. Then $h_2 = h_1$ on $N' \cap S^+$.

Choose $B > 0$ satisfying $B^{-1}\mathrm{diam}(S) \leq \mathrm{diam}\, h_i(S) \leq B\, \mathrm{diam}(S)$ for any subset $S \subset N'$ and $i = 1, 2$. Pick $a > 0$ so that $Q = \{(x, y) | \, |x| \leq a, 0 \leq y \leq 1\}$

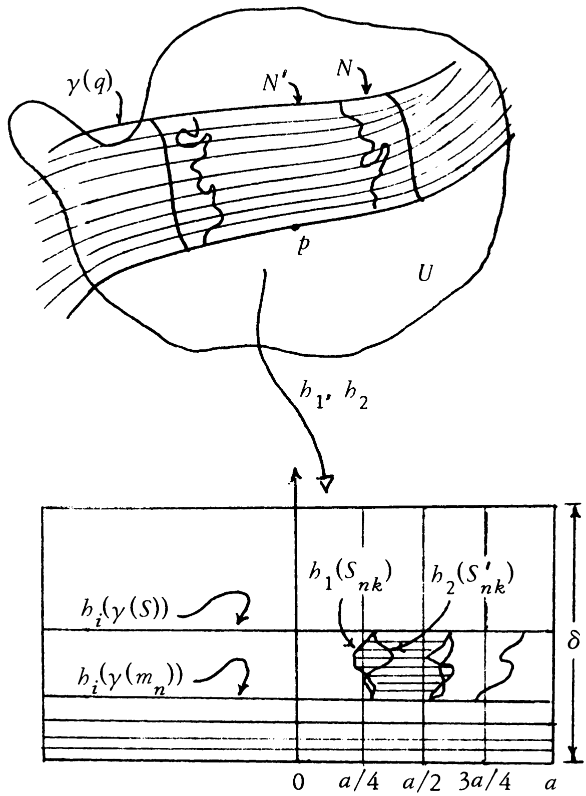

Figure 4

is contained in both $h_1(N')$, $h_2(N')$. Set $m = \min\{\epsilon, a/4\}$. $Q \cap h_i(S^+)$ contains segments arbitrarily close to 0. The complement of these consists of 'strips' which are the intersections of the images of the various half canonical regions with Q.

Now choose $\delta > 0$ so that the set $Q_\delta = \{(x, y) | \, |x| \leq a, \ 0 \leq y \leq \delta\}$ satisfies the following:

1. the supremum of widths of strips meeting Q_δ is less than $m/2B^2$;

2. the spacing between successive $h_i(p_{nk})$ along the image of any segment of a distinguished orbit $\gamma(m_n)$ which meets Q_δ is less than m.

Let S be a strip in Q_δ, bounded by segments of $h_i(\gamma(m_n))$ and $h_i(\gamma(s))$ (so $\gamma(s) \subset \delta R$ is a separatrix). Consider any of the points $h_i(p_{nk})$ lying between $x = -3a/4$ and $x = 3a/4$. There is a section across S, of both $h_1\phi_1$ and $h_2\phi_2$, with diameter less than $m/2B^2$. Its pre-image, under either h_i, has diameter less than $m/2B$. By our construction, both diam (S_{nk}),

$\operatorname{diam}(S'_{nk}) < m/B$, and, hence, both $\operatorname{diam}(h_1 S_{nk})$, $\operatorname{diam}(h_2 S'_{nk}) < \min\{\epsilon, a/4\}$. It follows that the rectangle $T = \{(x, y) \mid |x| \leq a/2,\ 0 \leq y \leq \delta\}$ is covered by cells (in either subdivision) of diameter less than 6ϵ. Each such cell intersects its image under the map, induced by h, $h_1 h h_1^{-1}: T \to T$, so that points close enough to p are moved an arbitrarily small distance by h.

Thus we have that h is a homeomorphism on the complement of the discrete set P of critical points in M. Furthermore, for any $p \in P$, there is a sequence $\{x_n\} \subset M - P$ with $x_n \to p$ and $h(x_n) \to p$. Any such homeomorphism extends by the identity to P.

(*Necessity*). By slightly modifying the argument given above, we may prove: If S_1^+, S_2^+ are two separatrix configurations for the same flow (M, ϕ), then there is a self-equivalence of (M, ϕ), taking S_1^+ onto S_2^+, and the identity on $S_1 = S_2$. It follows that any equivalence $(M, \phi_1) \to (M, \phi_2)$ induces an equivalence of the associated separatrix configurations.

BIBLIOGRAPHY

1. N. P. Bhatia and G. P. Szegö, *Stability theory of dynamical systems*, Die Grundlehren der math. Wissenschaften, Band 161, Springer-Verlag, New York and Berlin, 1970. MR 44 #7077.

2. O. Hájek, *Sections of dynamical systems in* E^2, Czechoslovak Math. J. 15 (90) (1965), 205–211. MR 31 #456.

3. L. Markus, *Global structure of ordinary differential equations in the plane*, Trans. Amer. Math. Soc. 76 (1954), 127–148. MR 15, 704.

DEPARTMENT OF MATHEMATICS, BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, OHIO 43403